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On The Numerical Eigenvalues of a Spring-Mass System

Jumah Aswad Zarnan[1], Wafaa Mustafa Hameed[2]

[1]*Asst. Prof. Dr., Dept. Of Accounting by IT, Cihan University \ Campus \ Sulaimaniya, Kurdistan Iraq*
[2]*Asst. Lect., Dept. Of Computer Science, Cihan University \ Campus \ Sulaimaniya, Kurdistan Iraq*
[1] jumah_223@yahoo.com; [2] afafa_ju223@yahoo.com

Abstract— In this paper we study eigenvalue problems [1]. Eigenvalues are special sets of scalars associated with a given matrix. In other words for a given matrixA, if there exist a non-zero vectorX, such that, $AX = \lambda X$ for scalar λ, then λ is called the eigenvalue of matrix A with corresponding eigenvector X. We consider here how to sole the systems of differential equations by reducing to solving eigenvalue problem.

Keywords— eigenvalues, eigenvectors, differential equations, matrix, reducing.

I. Introduction

Applications of eigenvalue problems play a great role in our real word. One class of applications which has recently gained considerable ground is that related to eigenvalues problems of a matrix. However, the most commonly solved eigenvalue problems today are those issues associated with the mass-spring system analysis of large structures. The mass-spring system frequencies are therefore determined by the eigenvalues of a square 3x3 matrix. This is an instance of simple eigenvalue problem $AX = \lambda X$ that is common in practice. Fortunately, numerical analysts have found an entirely different ways to calculate eigenvalues of a given square matrix. Among those methods QR method is the most widely used, important, accurate and speedy one. The primary reason that modern implementations of this method are efficient and reliable is that a QR factorization can be used to create each new matrix in the sequence and each QR factorization can be calculated quickly and accurately; it yields easily a new matrix orthogonally similar to the original matrix; and orthogonal similarities tend to minimize the effect of round off error on the eigenvalues [2].

II. PROBLEM DEFINITION

We consider here how to solve linear second order homogeneous systems of equations. Hence in this section we are interested in finding the general solution to a system of equations of the form [3].

$$\frac{d^2y}{dt^2} = AY \qquad (1)$$

Where A is an $n \times n$ constant matrix and the unknown $n \times 1$ solution vector Y consists of the components $y1, y2, \ldots, yn$. To solve such a system of equations we use the tried and trusted method of looking for solutions of the form:

$$y(t) = Ce^{\lambda t}$$

Where now C is a constant $n \times 1$ vector with components $c1, c2, \ldots, cn$. Substituting this form of the solution into the differential equation system (1), we get

$$\frac{d^2}{dt^2}\left(Ce^{\lambda t}\right) = ACe^{\lambda t}$$

$$\Leftrightarrow \quad \lambda^2 Ce^{\lambda t} = ACe^{\lambda t}$$

$$\Leftrightarrow \quad \lambda^2 C = AC$$

$$\Leftrightarrow \quad (A - \lambda^2 I)C = 0 \qquad (2)$$

Thus solving the system of differential equations (1) reduces to solving the eigenvalue problem (2)—though note that it contains λ^2 rather than λ. This is a standard eigenvalue problem. To avoid getting too technical here we will explore the method of solution and practical interpretations of the solution through the following illustrative example.

III. EXAMPLE (MASS-SPRING SYSTEM)

Three railway cars of mass $m1 = m3 = 750Kg, m2 = 500Kg$ move along a track and are connected by two buffer springs as shown in Figure 1. The springs have stiffness constants $k1 = k2 = 3000Kg/m$.

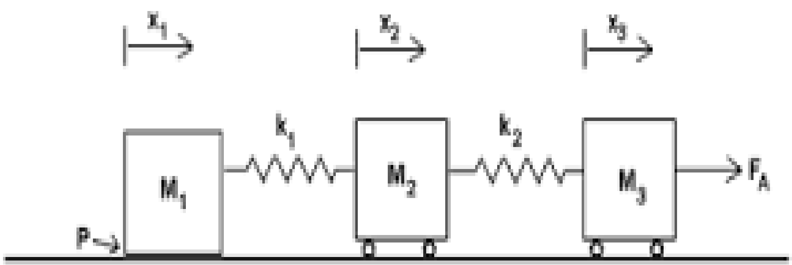

Fig1. Simple mass-spring three particle system.

Applying Newton's second law and Hooke's law, this mass-spring system gives rise to the differential equation system

$$\frac{d^2y}{dt^2} = AY \qquad (3)$$

Where A is the matrix

$$A = \begin{bmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{bmatrix} \qquad (4)$$

and

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is the vector of unknown position displacements (from equilibrium) for each of the masses shown in Figure 1.

By looking for a solution of the form

$$y(t) = c\, e^{\lambda t}$$

for a constant vector c, solution of the system of differential equations (3) reduces to solving the eigenvalue problem

$$(A - \lambda^2 I)c = 0 \qquad (5)$$

We know that the exact solutions to this eigenvalue problem are $\lambda 1^2 = 0$, $\lambda 2^2 = -4$ and $\lambda 3^2 = -16$. Writing $\lambda = i\omega$ so that $\lambda^2 = -\omega^2$, deduce the fundamental frequencies of oscillation $\omega 1, \omega 2$ and $\omega 3$ of the mechanical system in Figure 1. For each fundamental frequency of oscillation $\omega 1, \omega 2$ and $\omega 3$ corresponding to $\lambda 1, \lambda 2$ and $\lambda 3$, are the eigenvectors represent the possible modes of oscillation. Use those eigenvectors to enumerate the possible modes of oscillation of the masses corresponding to each eigenvalue–eigenvector pair.

The basic QR algorithm

In 1958 Rutishauser [4] of ETH Zurich experimented with a similar algorithm that we are going to present, but based on the LR factorization, i.e., based on Gaussian elimination without pivoting. That algorithm was not successful as the LR factorization (nowadays called LU factorization) is not stable without pivoting. Francis [5] noticed that the QR factorization would be the preferred choice and devised the QR algorithm with many of the bells and whistles used nowadays. Before presenting the complete picture, we start with a basic iteration, given in Algorithm 1, discuss its properties and improve on it step by step until we arrive at Francis' algorithm. We notice first that

$$A_k = R_k Q_k = Q_k^* A_{k-1} Q_k \qquad (5)$$

and hence A_k and A_{k-1} are unitarily similar. The matrix sequence $\{A_k\}$ converges (under certain assumptions) towards an upper triangular matrix [6]. Let us assume that the

Algorithm 1: Basic QR algorithm

1: $Let\ A \in Cn \times n. This\ algorithm\ computes\ an\ upper\ triangular\ matrix\ T\ and\ a\ unitary\ matrix\ U\ such\ that\ A = UTU^*\ is\ the\ Schur\ decomposition\ of\ A.$
2: $Set\ A_0 := A\ and\ U_0 = I.$
3: $for\ k = 1, 2, \ldots do$
4: $A_{k-1} =: Q_k R_k; /*\ QR\ factorization\ */$
5: $A_k := R_k Q_k;$
6: $U_k := U_{k-1} Q_k; /*\ Update\ transformation\ matrix\ */$
7: $end\ for$
8: $Set\ T := A_\infty\ and\ U := U_\infty.$

eigenvalues are mutually different in magnitude and we can therefore number the eigenvalues such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then the elements of A_k below the diagonal converge to zero like

$$\left|a_{ij}^{(k)}\right| = O\left(\left|\frac{\lambda_i}{\lambda_j}\right|^k\right), \qquad i > j. \qquad (6)$$

From (5) we see that

$$A_k = Q_k^* A_{k-1} Q_k = Q_k^* Q_{k-1}^* A_{k-2} Q_{k-1} Q_k = \cdots = Q_k^* \cdots Q_1^* A_0 \underbrace{Q_1 \cdots Q_k}_{U_k} \qquad (7)$$

With the same assumption on the eigenvalues, A_k tends to an upper triangular matrix and U_k converges to the matrix of Schur vectors.

Now, let us begin our discussion by using the QR algorithm for finding the eigenvalues of the above example.

There is no simple way to calculate eigenvalues for a matrix larger than 2x2 dimensions. The method of calculating the characteristic polynomial and then finding its zeros is not good numerically and moreover finding the roots of characteristic polynomial involves taking a determinant which uses large amount of computing time. The QR method is one of the most important methods which used to find eigenvalues of real square matrix. Therefore, the main idea of this section is determining all eigenvalues of real square matrix by using QR factorization (where Q is orthogonal and R is upper triangular matrices). To this end, suppose a real square A is given. Let $A = Q_0R_0$ be QR factorization of A (where Q is orthogonal and R is upper triangular matrices), and create $A_1 = R_0Q_0$. $Let\ A_1 = Q_1R_1$ be QR factorization of A_1 and similarly create $A_2 = R_1Q_1$, continue this process in the same fashion for $m \geq 1, (withh\ A_1 = A)$. Once A_m has been created such that, $A_m = Q_mR_m \quad and\ A_{m+1} = R_mQ_m$ Thus, the sequence $\{A_m\}$ will usually converges to something from which the eigenvalues can be computed easily. Moreover, A_2 is similar to A and A_{m+1} is similar to A_m and so on. Hence $A_m\ and\ A_{m+1}$ have the same eigenvalues as matrix A by definition of similarity of matrices. This process will be stopped when the entries below the main diagonal of current matrix Am are sufficiently small, or if it appears that convergence will not happen. This implies that, QR factorization sequence process can fail to converge or the convergence can be also extremely slow and expensive. If it converges to certain matrix, then the diagonal entries of this current matrix are tends to be eigenvalues of matrix A, if not, it can be modified in order to speed up convergence or to accelerate the rate of convergence of the given real square matrix dramatically we use methods like shifting of origin. The method of orthogonal triangularization (orthogonal transformation) is analogous to Gaussian elimination. Since orthogonal transformation will not worsen the condition or stability of eigenvalues of a non-symmetric matrix we will attempt to decompose an arbitrary real square matrix A into a product QR, where Q is orthogonal and R is upper triangular matrices.

Using MATLAB program we take the approximated eigenvalues and compare those with exact eigenvalues of the same given square matrix A of the example above, the results shown in the table I below.

TABLE I

ERRORS ANALYSIS OF EIGENVALUES WITH QR METHOD

Exact Eigenvalue	Approximate Eigenvalue	error
0	0	0.00E+00
-4	-4.0005	5.00E-04
-16	-15.9995	5.00E-04

IV. CONCLUSIONS

We have seen that calculating eigenvalues a matrix is an important problem in mathematics and the science but the native approach of solving characteristic polynomial is inefficient for large dimensional matrices. Rather, using the elegant QR methods provides an effective answer. The QR method is one of the most wildly used methods which used to find all eigenvalues of square matrix.

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