Available Online at www.ijcsmc.com

International Journal of Computer Science and Mobile Computing

A Monthly Journal of Computer Science and Information Technology

ISSN 2320–088X

IJCSMC, Vol. 2, Issue. 6, June 2013, pg.389 – 393

RESEARCH ARTICLE

Solving Improper Integrals with Maple

Chii-Huei Yu

Department of Management and Information, Nan Jeon Institute of Technology, Tainan City, Taiwan
chiihuei@mail.njtc.edu.tw

Abstract—This study takes the mathematical software Maple as the auxiliary tool to evaluate some type of improper integrals. We can obtain the infinite series form of this type of improper integrals by using three important methods (i.e., differentiation with respect to a parameter, differentiation term by term, and integration term by term). In addition, we propose two improper integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

Keywords—improper integrals, infinite series form, differentiation with respect to a parameter, differentiation term by term, integration term by term, Maple

I. INTRODUCTION

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, [1]-[7] can be adopted as references.

In mathematics, physics and engineering curricula, many important special functions can be represented as improper integrals, for example, the Gamma function and the Beta function. Therefore, the evaluation and numerical calculation of improper integrals possess significance, and can be studied based on [8]-[13]. In this paper, we mainly study the following type of improper integral

$$\int_0^1 (\ln x)^m x^a \cdot f(x)dx \qquad (1)$$

, where a is a real number, $a > 0$, m is a positive integer, and $f(x)$ is a rational function. Suppose $f(x)$ can be expressed by

$$f(x) = \frac{B_1}{x+r_1} + \frac{B_2}{x+r_2} + \cdots + \frac{B_p}{x+r_p} \tag{2}$$

, where p is a positive integer, and r_i, B_i are real numbers, $|r_i| > 1$ for all $i = 1,..., p$. We can obtain the infinite series forms of this type of improper integral by using three important methods: differentiation with respect to a parameter, differentiation term by term, and integration term by term; this is the major result in this study (i.e., Theorem A). In addition, we obtain a corollary from this theorem (Corollary A). On the other hand, we provide two improper integrals to determine their infinite series forms practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

II. MAIN RESULTS

Firstly, we introduce a formula used in this study.

Geometric series .

Assume u is a real number, $|u| < 1$. Then $\frac{1}{1+u} = \sum_{n=0}^{\infty} (-1)^n u^n$.

Next, we introduce three important theorems used in this paper.

Differentiation with respect to a parameter ([14]).

Suppose the two-variables function $f(\lambda, x)$ is defined on $[\lambda_1, \lambda_2] \times [r, s]$. If $f(\lambda, x)$ and its partial derivative $\frac{\partial f}{\partial \lambda}(\lambda, x)$ are continuous functions on $[\lambda_1, \lambda_2] \times [r, s]$. Then $F(\lambda) = \int_r^s f(\lambda, x)dx$ is differentiable on (λ_1, λ_2) , and its derivative $\frac{d}{d\lambda}F(\lambda) = \int_r^s \frac{\partial f}{\partial \lambda}(\lambda, x)dx$ for $\lambda \in (\lambda_1, \lambda_2)$.

Differentiation term by term ([15, p230]).

If, for all non-negative integer k , the functions $g_k : (a,b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b) , (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b) . Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b) . Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

Integration term by term ([15, p269]).

Suppose $\{g_n\}_{n=0}^{\infty}$ is a sequence of Lebesgue integrable functions defined on an inteval I . If $\sum_{n=0}^{\infty} \int_I |g_n|$ is convergent, then $\int_I \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} \int_I g_n$.

The following is the major result in this study, we obtain the infinite series forms of improper integral (1).

Theorem A. Suppose m, p are positive integers, a is a real number, $a > 0$, and r_i, B_i are real numbers, $|r_i| > 1$ for all $i = 1,..., p$. Let $f(x)$ be a rational function which can be expressed by

$$f(x) = \frac{B_1}{x+r_1} + \frac{B_2}{x+r_2} + \cdots + \frac{B_p}{x+r_p}$$

Then the improper integral

$$\int_0^1 (\ln x)^m x^a \cdot f(x)dx = (-1)^m m! \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a+1)^{m+1}} \cdot \left(\frac{B_1}{r_1^{n+1}} + \frac{B_2}{r_2^{n+1}} + \cdots + \frac{B_p}{r_p^{n+1}} \right) \quad (3)$$

Proof. $f(x)$

$$= \frac{B_1}{x+r_1} + \frac{B_2}{x+r_2} + \cdots + \frac{B_p}{x+r_p}$$

$$= \frac{B_1}{r_1} \cdot \frac{1}{1+\frac{x}{r_1}} + \frac{B_2}{r_2} \cdot \frac{1}{1+\frac{x}{r_2}} + \cdots + \frac{B_p}{r_p} \cdot \frac{1}{1+\frac{x}{r_p}}$$

$$= \frac{B_1}{r_1} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{r_1}\right)^n + \frac{B_2}{r_2} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{r_2}\right)^n + \cdots + \frac{B_p}{r_p} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{r_p}\right)^n \quad \text{(by geometric series)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{B_1}{r_1^{n+1}} + \frac{B_2}{r_2^{n+1}} + \cdots + \frac{B_p}{r_p^{n+1}} \right) \cdot x^n \quad (4)$$

Therefore,

$$\int_0^1 x^a \cdot f(x)dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \left(\frac{B_1}{r_1^{n+1}} + \frac{B_2}{r_2^{n+1}} + \cdots + \frac{B_p}{r_p^{n+1}} \right) \cdot x^{n+a} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{B_1}{r_1^{n+1}} + \frac{B_2}{r_2^{n+1}} + \cdots + \frac{B_p}{r_p^{n+1}} \right) \cdot \int_0^1 x^{n+a} dx \quad \text{(using integration term by term)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+a+1} \cdot \left(\frac{B_1}{r_1^{n+1}} + \frac{B_2}{r_2^{n+1}} + \cdots + \frac{B_p}{r_p^{n+1}} \right) \quad (5)$$

By differentiation with respect to a parameter and differentiation term by term, differentiating m times with respect to a on both sides of (5), we obtain

$$\int_0^1 (\ln x)^m x^a \cdot f(x)dx = (-1)^m m! \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a+1)^{m+1}} \cdot \left(\frac{B_1}{r_1^{n+1}} + \frac{B_2}{r_2^{n+1}} + \cdots + \frac{B_p}{r_p^{n+1}} \right) \quad \blacksquare$$

In Theorem A, substituting $x = e^{-t}$, we immediately have the following result.

Corollary A. If the assumptions are the same as Theorem A. Then the improper integral

$$\int_0^{\infty} t^m e^{-(a+1)t} f(e^{-t})dt = m! \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a+1)^{m+1}} \cdot \left(\frac{B_1}{r_1^{n+1}} + \frac{B_2}{r_2^{n+1}} + \cdots + \frac{B_p}{r_p^{n+1}} \right) \quad (6)$$

III. EXAMPLES

In the following, we provide two improper integrals to determine their infinite series forms practically. On the other hand, we use Maple to calculate the approximations of these improper integrals and their infinite series forms for verifying our answers.

Example 1. Because $\dfrac{9x^2+55x+86}{x^3+9x^2+26x+24}=\dfrac{9x^2+55x+86}{(x+2)(x+3)(x+4)}$

$$=\frac{6}{x+2}+\frac{-2}{x+3}+\frac{5}{x+4} \tag{7}$$

Using Theorem A, we can determine the following improper integral

$$\int_0^1 \frac{(\ln x)^3 x^{2/5}(9x^2+55x+86)}{x^3+9x^2+26x+24}dx = -6\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+7/5)^4}\cdot\left(\frac{6}{2^{n+1}}+\frac{-2}{3^{n+1}}+\frac{5}{4^{n+1}}\right) \tag{8}$$

We employ Maple to verify the correctness of (8) as follows:

```
>evalf(int((ln(x))^3*x^(2/5)*(9*x^2+55*x+86)/(x^3+9*x^2+26*x+24),x=0..1),19);
```

−5.338009293813000262

```
>evalf(-6*sum((-1)^n/(n+7/5)^4*(6/2^(n+1)-2/3^(n+1)+5/4^(n+1)),n=0..infinity),22);
```

−5.338009293813000260 + 0. I

The above answer obtained by Maple appears I ($=\sqrt{-1}$), it is because Maple calculates by using special functions built in. The imaginary part is 0, so can be ignored.

Example 2. Because $\dfrac{-2e^{-2t}+22e^{-t}-8}{e^{-3t}-19e^{-t}+30}=\dfrac{-2e^{-2t}+22e^{-t}-8}{(e^{-t}-2)(e^{-t}-3)(e^{-t}+5)}$

$$=\frac{-4}{e^{-t}-2}+\frac{5}{e^{-t}-3}+\frac{-3}{e^{-t}+5} \tag{9}$$

By Corollary A, we obtain the following improper integral

$$\int_0^{\infty} \frac{t^7 e^{-(7/3)t}(-2e^{-2t}+22e^{-t}-8)}{e^{-3t}-19e^{-t}+30}dt = 5040\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+7/3)^8}\cdot\left(\frac{-4}{(-2)^{n+1}}+\frac{5}{(-3)^{n+1}}+\frac{-3}{5^{n+1}}\right) \tag{10}$$

In the following, we also use Maple to verify the correctness of (10).

```
>evalf(int(t^7*exp(-7/3*t)*(-2*exp(-2*t)+22*exp(-t)-8)/(exp(-3*t)-19*exp(-t)+30),t=0..infinity),18);
```

−1.32945937027594590

```
>evalf(5040*sum((-1)^n/(n+7/3)^8*(-4/(-2)^(n+1)+5/(-3)^(n+1)-3/5^(n+1)),n=0..infinity),26);
```

−1.32945937027594589 + 0. I

The above answer also appears I, but the imaginary part is 0, so can be ignored.

IV. CONCLUSIONS

As mentioned, the three important methods (i.e., differentiation with respect to a parameter, differentiation term by term, and integration term by term) play significant roles in the theoretical inferences of this study. In fact, the applications of these theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications.

On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

References

[1] F. Garvan, *The Maple Book*, London: Chapman & Hall/CRC, 2001.
[2] M. L. Abell and J. P. Braselton, *Maple by Example*, 3rd ed., New York: Elsevier Academic Press, 2005.
[3] R. J. Stroeker and J. F. Kaashoek, *Discovering Mathematics with Maple* : *An Interactive Exploration for Mathematicians, Engineers and Econometricians*, Basel: Birkhauser Verlag, 1999.
[4] C. T. J. Dodson and E. A. Gonzalez, *Experiments in Mathematics Using Maple*, New York: Springer-Verlag, 1995.
[5] C. Tocci and S. G. Adams, *Applied Maple for Engineers and Scientists*, Boston: Artech House, 1996.
[6] D. Richards, *Advanced Mathematical Methods with Maple*, New York: Cambridge University Press, 2002.
[7] J. S. Robertson, *Engineering Mathematics with Maple*, New York: McGraw-Hill, 1996.
[8] C.-H. Yu, " Power series forms of four types of improper integrals,"*Chna Nan Annual Bulletin*, vol. 38, in press.
[9] C.-H. Yu, "The improper integral problems related to hyperbolic functions," *Journal of Tzu Hui* , vol. 8, pp. 192-203, 2012.
[10] C.-H. Yu, "Generalization of some improper integral formulas,"*Cheng Shiu Journal of General Education*, vol. 10, in press.
[11] C.-H. Yu, "The series expressions of some integrals," *Yu Da Academic Journal*, vol. 32, pp.93-103, 2012.
[12] C.-H. Yu, "Application of Maple: taking two types of improper integral problems as examples," *Proceedings of 2013 Information Technology and Management Symposium*, B9-B14, Hwa Hsia Institute of Technology, Taiwan, March 2013.
[13] C.-H. Yu, "Application of Maple on some type of improper integral problem," *Proceedings of 2012 Workshop on Consumer Electronics*, P0081, National Formosa University, Taiwan, November 2012.
[14] L. Flatto, *Advanced Calculus*, Baltimore: The Williams & Wilkins, p405, 1976.
[15] T. M. Apostol, *Mathematical Analysis*, 2nd ed., Boston: Addison-Wesley, 1975.

