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RESEARCH ARTICLE

A Study on the Differential Problems Using Maple

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Abstract—This paper uses the mathematical software Maple for the auxiliary tool to study the differential problems of two types of functions. We can obtain the closed forms of any order derivatives of these two types of functions mainly using binomial theorem and Leibniz differential rule. In addition, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

Keywords—derivatives, closed forms, binomial theorem, Leibniz differential rule, Maple

I. INTRODUCTION

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. As for the instructions and operations of Maple, we can refer to [1]-[7].

In calculus and engineering mathematics curricula, finding $f^{(n)}(c)$ (the n-th order derivative value of function $f(x)$ at $x=c$), in general, necessary goes through two procedures: Firstly evaluating $f^{(n)}(x)$ (the n-th order derivative of $f(x)$), and secondly substituting $x=c$ to $f^{(n)}(x)$. When evaluating the higher order derivative values of a function (i.e. n is large), these two procedures will make us face with increasingly complex calculations. Therefore, to obtain the answers through manual calculations is not an easy thing. In this paper, we mainly study the evaluation of derivatives of the following two types of functions

$$f(x) = x^r \cos^p(ax+b) \tag{1}$$

$$g(x) = x^r \sin^p(ax+b) \tag{2}$$

, where r, a, b are real numbers, p is a positive integer. We can obtain the closed forms of any order derivatives of these two types of functions mainly using binomial theorem and Leibniz differential rule; these are the major results in this paper (i.e., Theorems 1 and 2), and hence greatly reduce the difficulty of determining higher order derivatives values of these two types of functions. As for the related study of differential problems can refer to [8]-[15]. On the other hand, we propose two functions to determine their derivatives and calculate some of their higher order derivative values practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

II. MAIN RESULTS

Firstly, we introduce some notations and formulas used in this study.

Notations.

(i)Suppose t is any real number, and m is any positive integer. Define $(t)_m = t(t-1)\cdots(t-m+1)$, and $(t)_0 = 1$.

(ii) Assume s is any real number, the largest integer less than or equal to s , is denoted by $\lfloor s \rfloor$.

Formulas.

(i)*Euler's formula*.

$e^{i\theta} = \cos\theta + i\sin\theta$, where θ is any real number.

(ii)*DeMoivre's formula*.

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is any integer, θ is any real number.

Next, we introduce two important theorems used in this paper.

Binomial theorem. Suppose u, v are real numbers, p is a positive integer. Then $(u+v)^p = \sum_{k=0}^{p} \binom{p}{k} u^{p-k} v^k$.

Leibniz differential rule ([16]). Let n be a positive integer, and $f(x), g(x)$ are functions such that their m-th order derivatives $f^{(m)}(x), g^{(m)}(x)$ exist for all $m = 1,...,n$. Then the n-th order derivative of product function $f(x)g(x)$,

$$(fg)^{(n)}(x) = \sum_{m=0}^{n} \binom{n}{m} f^{(n-m)}(x) g^{(m)}(x)$$

, where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

The following is the first result in this study, we determine the closed forms of any order derivatives of function (1).

Theorem 1. Suppose r, a, b are real numbers, p, n are positive integers. If the domain of the function

$$f(x) = x^r \cos^p(ax+b)$$

is $\{x \in R \mid x^r \, exist \, , x \neq 0\}$, then the n-th order derivative of $f(x)$,

$$f^{(n)}(x) = \frac{1}{2^p} \cdot \sum_{m=0}^{n} \sum_{k=0}^{p} \binom{n}{m} \binom{p}{k} a^m (p-2k)^m (r)_{n-m} \cdot x^{r-n+m} \cdot \cos\left[(p-2k)(ax+b) + \frac{m\pi}{2}\right]$$

$$+\frac{1+(-1)^p}{2}\cdot\frac{1}{2^p}\cdot\binom{p}{\lfloor p/2\rfloor}(r)_n x^{r-n} \quad (3)$$

for all x satisfy x^r exist, $x \neq 0$.

Proof. $f(x) = x^r \cos^p(ax+b)$

$$= x^r\left[\frac{1}{2}\left(e^{i(ax+b)}+e^{-i(ax+b)}\right)\right]^p \quad \text{(by Euler's formula)}$$

$$= \frac{1}{2^p}\cdot x^r\cdot\sum_{k=0}^{p}\binom{p}{k}[e^{i(ax+b)}]^{(p-k)}[e^{-i(ax+b)}]^{(k)} \quad \text{(by Binomial theorem)}$$

$$= \frac{1}{2^p}\cdot x^r\cdot\sum_{k=0}^{p}\binom{p}{k}e^{i(p-2k)(ax+b)} \quad \text{(by DeMoivre's formula)}$$

$$= \frac{1}{2^p}\cdot x^r\cdot\sum_{k=0}^{p}\binom{p}{k}\cos[(p-2k)(ax+b)] \quad \text{(by Euler's formula)}$$

$$= \frac{1}{2^{p-1}}\cdot x^r\cdot\sum_{k=0}^{\left\lfloor\frac{p-1}{2}\right\rfloor}\binom{p}{k}\cos[(p-2k)(ax+b)]+\frac{1+(-1)^p}{2}\cdot\frac{1}{2^p}\cdot\binom{p}{\lfloor p/2\rfloor}\cdot x^r \quad (4)$$

Therefore, the n -th order derivative of $f(x)$,

$f^{(n)}(x)$

$$= \frac{1}{2^p}\cdot\sum_{m=0}^{n}\binom{n}{m}(x^r)^{(n-m)}\cdot\left(\sum_{k=0}^{p}\binom{p}{k}\cos[(p-2k)(ax+b)]\right)^{(m)}+\frac{1+(-1)^p}{2}\cdot\frac{1}{2^p}\cdot\binom{p}{\lfloor p/2\rfloor}(r)_n x^{r-n}$$

(by Leibniz differential rule)

$$= \frac{1}{2^p}\cdot\sum_{m=0}^{n}\sum_{k=0}^{p}\binom{n}{m}\binom{p}{k}a^m(p-2k)^m(r)_{n-m}\cdot x^{r-n+m}\cdot\cos\left[(p-2k)(ax+b)+\frac{m\pi}{2}\right]$$

$$+\frac{1+(-1)^p}{2}\cdot\frac{1}{2^p}\cdot\binom{p}{\lfloor p/2\rfloor}(r)_n x^{r-n}$$

for all x satisfy x^r exist, $x \neq 0$ ■

Using Theorem 1, we immediately obtain the second result in this study, we find the closed forms of any order derivatives of function (2).

Theorem 2. If the assumptions are the same as Theorem 1, and the domain of the function

$$g(x) = x^r \sin^p(ax+b)$$

is $\{x \in R \,|\, x^r\, exist\,,\, x \neq 0\}$, then the n -th order derivative of $g(x)$,

$$g^{(n)}(x) = \frac{1}{2^p} \cdot \sum_{m=0}^{n} \sum_{k=0}^{p} \binom{n}{m} \binom{p}{k} a^m (p-2k)^m (r)_{n-m} \cdot x^{r-n+m} \cdot \cos\left[(p-2k)\left(ax+b-\frac{\pi}{2} \right) + \frac{m\pi}{2} \right]$$

$$+ \frac{1+(-1)^p}{2} \cdot \frac{1}{2^p} \cdot \binom{p}{\lfloor p/2 \rfloor} (r)_n x^{r-n} \tag{5}$$

for all x satisfy x^r exist, $x \neq 0$.

Proof. Because $g(x) = x^r \sin^p(ax+b) = x^r \cos^p\left(ax+b-\frac{\pi}{2}\right)$, by Theorem 1, we easily obtain the n-th order derivative of $g(x)$,

$$g^{(n)}(x) = \frac{1}{2^p} \cdot \sum_{m=0}^{n} \sum_{k=0}^{p} \binom{n}{m} \binom{p}{k} a^m (p-2k)^m (r)_{n-m} \cdot x^{r-n+m} \cdot \cos\left[(p-2k)\left(ax+b-\frac{\pi}{2} \right) + \frac{m\pi}{2} \right]$$

$$+ \frac{1+(-1)^p}{2} \cdot \frac{1}{2^p} \cdot \binom{p}{\lfloor p/2 \rfloor} (r)_n x^{r-n}$$

for all x satisfy x^r exist, $x \neq 0$ ■

III. EXAMPLES

In the following, we propose two functions to determine their any order derivatives and some of their higher order derivative values practically. On the other hand, we use Maple to calculate the approximations of these higher order derivative values and their closed forms for verifying our answers.

Example 1. If the domain of the function

$$f(x) = x^{3/4} \cos^8\left(2x - \frac{5\pi}{6}\right) \tag{6}$$

is $\{x \in R \mid x > 0\}$. Using Theorem 1, we obtain any n-th order derivative of $f(x)$,

$$f^{(n)}(x) = \frac{1}{2^8} \cdot \sum_{m=0}^{n} \sum_{k=0}^{8} \binom{n}{m} \binom{8}{k} 2^m (8-2k)^m (3/4)_{n-m} \cdot x^{3/4-n+m} \cdot \cos\left[(8-2k)\left(2x-\frac{5\pi}{6} \right) + \frac{m\pi}{2} \right]$$

$$+ \frac{1}{2^8} \cdot \binom{8}{4} (3/4)_n x^{3/4-n} \tag{7}$$

for all $x > 0$.

Thus, the 11 -th order derivative value of $f(x)$ at $x = \frac{9}{7}$,

$$f^{(11)}\left(\frac{9}{7}\right)$$

$$= \frac{1}{2^8} \cdot \sum_{m=0}^{11} \sum_{k=0}^{8} \binom{11}{m} \binom{8}{k} 2^m (8-2k)^m (3/4)_{11-m} \cdot \left(\frac{9}{7}\right)^{m-41/4} \cdot \cos\left[(8-2k)\left(\frac{18}{7}-\frac{5\pi}{6} \right) + \frac{m\pi}{2} \right]$$

$$+\frac{1}{2^8}\cdot\binom{8}{4}(3/4)_{11}\left(\frac{9}{7}\right)^{-41/4} \tag{8}$$

Next, we use Maple to verify the correctness of (8).

>f:=x->x^(3/4)*(cos(2*x-5*Pi/6))^8;

$$f := x \to x^{3/4}\cos\left(2\,x-\frac{5}{6}\,\pi\right)^8$$

>evalf((D@@11)(f)(9/7),18);

$$-1.69714344106967744\cdot 10^{11}$$

>evalf(1/2^8*sum(sum(11!/(m!*(11-m)!)*8!/(k!*(8-k)!)*2^m*(8-2*k)^m*product(3/4-j,j=0..(10-m))*(9/7)^(m-41/4)*cos((8-2*k)*(18/7-5*Pi/6)+m*Pi/2),k=0..8),m=0..11)+1/2^8*8!/(4!*4!)*product(3/4-j,j=0..10)*(9/7)^(-41/4),18);

$$-1.69714344106967736\cdot 10^{11}$$

Example 2. If the domain of the function

$$g(x) = x^{4/9}\sin^{11}\left(4x+\frac{7\pi}{8}\right) \tag{9}$$

is $\{x \in R \mid x \neq 0\}$. Using Theorem 2, we can determine any n -th order derivative of $g(x)$,

$$g^{(n)}(x)$$

$$=\frac{1}{2^{11}}\cdot\sum_{m=0}^{n}\sum_{k=0}^{11}\binom{n}{m}\binom{11}{k}4^m(11-2k)^m(4/9)_{n-m}\cdot x^{4/9-n+m}\cdot\cos\left[(11-2k)\left(4x+\frac{7\pi}{8}-\frac{\pi}{2}\right)+\frac{m\pi}{2}\right] \tag{10}$$

for all $x \neq 0$.

Therefore, the 14 -th order derivative value of $g(x)$ at $x=\frac{8}{13}$,

$$g^{(14)}\left(\frac{8}{13}\right)$$

$$=\frac{1}{2^{11}}\cdot\sum_{m=0}^{14}\sum_{k=0}^{11}\binom{14}{m}\binom{11}{k}4^m(11-2k)^m(4/9)_{14-m}\cdot\left(\frac{8}{13}\right)^{-122/9+m}\cdot\cos\left[(11-2k)\left(\frac{32}{13}+\frac{3\pi}{8}\right)+\frac{m\pi}{2}\right] \tag{11}$$

we also use Maple to verify the correctness of (11).

>g:=x->x^(4/9)*(sin(4*x+7*Pi/8))^11;

$$g := x \to x^{4/9}\sin\left(4\,x+\frac{7}{8}\,\pi\right)^{11}$$

>evalf((D@@14)(g)(8/13),26);

$$8.8250361726322581062437587\cdot 10^{18}$$

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>evalf(1/2^11*sum(sum(14!/(m!*(14-m)!)*11!/(k!*(11-k)!)*4^m*(11-2*k)^m*product(4/9-j,j=0..(13-m))*
(8/13)^(-122/9+m)*cos((11-2*k)*(32/13+3*Pi/8)+m*Pi/2),k=0..11),m=0..14),26);
```

$$8.8250361726322581062436259 \cdot 10^{18}$$

IV. CONCLUSIONS

As mentioned, the binomial theorem and the Leibniz differential rule play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications.

On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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