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RESEARCH ARTICLE

Evaluating Some Integrals with Maple

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Abstract*—This paper uses the mathematical software Maple for the auxiliary tool to evaluate two types of integrals. We can obtain the closed forms of these two types of integrals by using Euler's formula, DeMoivre's formula, and finite geometric series. In addition, we provide two integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

***Keywords*— integrals; closed forms; Euler's formula; DeMoivre's formula; finite geometric series; Maple**

I. Introduction

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, we can refer to [1]-[7].

In calculus courses, we learnt many methods to solve the integral problems, including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, and so on. This paper mainly studies the following two types of integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int \frac{1-e^{ax}\cos(bx+c)-e^{a(n+1)x}\cos(n+1)(bx+c)+e^{a(n+2)x}\cos n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}}dx \qquad (1)$$

$$\int \frac{e^{ax}\sin(bx+c)-e^{a(n+1)x}\sin(n+1)(bx+c)+e^{a(n+2)x}\sin n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}}dx \qquad (2)$$

, where n is any non-negative integer, a,b,c are real numbers, $a^2+b^2\neq 0$, and $(ax)^2+(bx+c)^2\neq 0$. We can obtain the closed forms of these two types of integrals by using Euler's formula, DeMoivre's formula, and finite geometric series ; these are the major results in this study (i.e., Theorems 1, 2). As for the study of related integral problems can refer to [8]-[14]. On the other hand, we provide two integrals to determine their closed forms practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

II. MAIN RESULTS

Firstly, we introduce a notation and some formulas used in this study.

Notation.

Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, a, b are real numbers. We denote a the real part of z by $\mathrm{Re}(z)$, and b the imaginary part of z by $\mathrm{Im}(z)$.

Euler's formula.

$e^{iy} = \cos y + i \sin y$, where y is any real number.

DeMoivre's formula.

$(\cos y + i \sin y)^n = \cos ny + i \sin ny$, where n is any integer, y is any real number.

Finite geometric series.

$1 + z + z^2 + \cdots + z^n = \dfrac{1 - z^{n+1}}{1 - z}$, where z is a complex number, $z \neq 1$ and n is any non-negative integer.

Before deriving the major results in this study, we need a lemma.

Lemma A. Assume p, r, s are real numbers, and $p^2 + r^2 \neq 0$, C is a constant. Then the indefinite integrals

$$\int e^{px} \cos(rx + s)dx = \frac{1}{p^2 + r^2} e^{px} [p \cos(rx + s) + r \sin(rx + s)] + C \tag{3}$$

$$\int e^{px} \sin(rx + s)dx = \frac{1}{p^2 + r^2} e^{px} [p \sin(rx + s) - r \cos(rx + s)] + C \tag{4}$$

Proof.

$$\int e^{px} \cos(rx + s)dx$$

$$= \int \mathrm{Re}[e^{px + i(rx+s)}]dx \quad \text{(by Euler's formula)}$$

$$= \mathrm{Re}[\int e^{(p+ir)x + is} dx]$$

$$= \mathrm{Re}\left(\frac{1}{p + ir} e^{(p+ir)x+is}\right) + C$$

$$= \mathrm{Re}\left(\frac{p - ir}{p^2 + r^2} e^{px} [\cos(rx + s) + i \sin(rx + s)]\right) + C$$

$$= \frac{1}{p^2 + r^2} e^{px} [p \cos(rx + s) + r \sin(rx + s)] + C$$

Similarly,

$$\int e^{px}\sin(rx+s)dx$$

$$=\int \mathrm{Im}[e^{px+i(rx+s)}]dx \qquad \text{(by Euler's formula)}$$

$$=\mathrm{Im}\left(\frac{1}{p+ir}e^{(p+ir)x+is}\right)+C$$

$$=\mathrm{Im}\left(\frac{p-ir}{p^2+r^2}e^{px}[\cos(rx+s)+i\sin(rx+s)]\right)+C$$

$$=\frac{1}{p^2+r^2}e^{px}[p\sin(rx+s)-r\cos(rx+s)]+C$$ ■

The following is the first main result in this paper, we obtain the closed form of indefinite integral (1).

Theorem 1. Suppose n is any non-negative integer, a,b,c are real numbers, $a^2+b^2\neq 0$, C is a constant, and $(ax)^2+(bx+c)^2\neq 0$. Then the indefinite integral

$$\int\frac{1-e^{ax}\cos(bx+c)-e^{a(n+1)x}\cos(n+1)(bx+c)+e^{a(n+2)x}\cos n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}}dx$$

$$=x+\frac{1}{a^2+b^2}\cdot\sum_{k=1}^{n}\frac{1}{k}e^{akx}[a\cos k(bx+c)+b\sin k(bx+c)]+C \qquad (5)$$

Proof. Let $z=e^{ax+i(bx+c)}$, then by finite geometric series

$$\frac{1-z^{n+1}}{1-z}=1+z+z^2+\cdots+z^n$$

$$\Rightarrow \frac{1-[e^{ax+i(bx+c)}]^{n+1}}{1-e^{ax+i(bx+c)}}=\sum_{k=0}^{n}[e^{ax+i(bx+c)}]^k$$

$$\Rightarrow \frac{1-e^{a(n+1)x+i(n+1)(bx+c)}}{1-e^{ax}\cdot e^{i(bx+c)}}=\sum_{k=0}^{n}e^{akx+ik(bx+c)} \qquad \text{(by DeMoivre's formula)}$$

$$\Rightarrow \frac{1-e^{a(n+1)x}\cdot[\cos(n+1)(bx+c)+i\sin(n+1)(bx+c)]}{1-e^{ax}[\cos(bx+c)+i\sin(bx+c)]}=\sum_{k=0}^{n}e^{akx}\cdot[\cos k(bx+c)+i\sin k(bx+c)]$$

(by Euler's formula)

$$\Rightarrow \frac{\{[1-e^{a(n+1)x}\cos(n+1)(bx+c)]-ie^{a(n+1)x}\sin(n+1)(bx+c)\}\{[1-e^{ax}\cos(bx+c)]+ie^{ax}\sin(bx+c)\}}{[1-e^{ax}\cos(bx+c)]^2+e^{2ax}\sin^2(bx+c)}$$

$$=\sum_{k=0}^{n}[e^{akx}\cos k(bx+c)+ie^{akx}\sin k(bx+c)] \qquad (6)$$

Using the equal of the real parts of both sides of (6), we obtain

$$\frac{1-e^{ax}\cos(bx+c)-e^{a(n+1)x}\cos(n+1)(bx+c)+e^{a(n+2)x}\cos n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}} = \sum_{k=0}^{n} e^{akx}\cos k(bx+c) \qquad (7)$$

Therefore, the indefinite integral

$$\int \frac{1-e^{ax}\cos(bx+c)-e^{a(n+1)x}\cos(n+1)(bx+c)+e^{a(n+2)x}\cos n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}}dx$$

$$=\int \sum_{k=0}^{n} e^{akx}\cos k(bx+c)dx$$

$$=\sum_{k=0}^{n}\int e^{akx}\cos k(bx+c)dx$$

$$= x+\frac{1}{a^2+b^2}\cdot\sum_{k=1}^{n}\frac{1}{k}e^{akx}[a\cos k(bx+c)+b\sin k(bx+c)]+C \qquad \text{(by (3) of Lemma A)}$$ ■

Next, we derive the second major result in this study, we determine the closed form of indefinite integral (2).

Theorem 2. If the assumptions are the same as Theorem 1. Then the indefinite integral

$$\int \frac{e^{ax}\sin(bx+c)-e^{a(n+1)x}\sin(n+1)(bx+c)+e^{a(n+2)x}\sin n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}}dx$$

$$=\frac{1}{a^2+b^2}\cdot\sum_{k=1}^{n}\frac{1}{k}e^{akx}[a\sin k(bx+c)-b\cos k(bx+c)]+C \qquad (8)$$

Proof. By the equal of the imaginary parts of both sides of (6), we obtain

$$\frac{e^{ax}\sin(bx+c)-e^{a(n+1)x}\sin(n+1)(bx+c)+e^{a(n+2)x}\sin n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}} = \sum_{k=0}^{n} e^{akx}\sin k(bx+c) \qquad (9)$$

Thus, the indefinite integral

$$\int \frac{e^{ax}\sin(bx+c)-e^{a(n+1)x}\sin(n+1)(bx+c)+e^{a(n+2)x}\sin n(bx+c)}{1-2e^{ax}\cos(bx+c)+e^{2ax}}dx$$

$$=\int \sum_{k=0}^{n} e^{akx}\sin k(bx+c)dx$$

$$=\sum_{k=1}^{n}\int e^{akx}\sin k(bx+c)dx$$

$$=\frac{1}{a^2+b^2}\cdot\sum_{k=1}^{n}\frac{1}{k}e^{akx}[a\sin k(bx+c)-b\cos k(bx+c)]+C \qquad \text{(by (4) of Lemma A)}$$ ■

III. EXAMPLES

In the following, we propose two indefinite integrals and determine their closed forms practically. In addition, we evaluate some related definite integrals and use Maple to calculate the approximations of these definite integrals and their closed forms for verifying our answers.

Example 1. By Theorem 1, we obtain the following indefine integral

$$\int \frac{1-e^{2x}\cos(4x+3)-e^{12x}\cos 6(4x+3)+e^{14x}\cos 5(4x+3)}{1-2e^{2x}\cos(4x+3)+e^{4x}}dx$$

$$= x+\frac{1}{20}\cdot\sum_{k=1}^{5}\frac{1}{k}e^{2kx}[2\cos k(4x+3)+4\sin k(4x+3)]+C \tag{10}$$

for all $x \in R$.

Therefore, we can determine the following definite integral

$$\int_{-10}^{-3} \frac{1-e^{2x}\cos(4x+3)-e^{12x}\cos 6(4x+3)+e^{14x}\cos 5(4x+3)}{1-2e^{2x}\cos(4x+3)+e^{4x}}dx$$

$$= 7+\frac{1}{20}\cdot\sum_{k=1}^{5}\frac{1}{k}e^{-6k}(2\cos 9k-4\sin 9k)-\frac{1}{20}\cdot\sum_{k=1}^{5}\frac{1}{k}e^{-20k}(2\cos 37k-4\sin 37k) \tag{11}$$

Next, we employ Maple to verify the correctness of (11) .

```
>evalf(int((1-exp(2*x)*cos(4*x+3)-exp(12*x)*cos(24*x+18)+exp(14*x)*cos(20*x+15))/(1-2*exp(2*x)*cos
(4*x+3)+exp(4*x)),x=-10..-3),24);
```

$$6.99957050820771214882333 - 7.96935786151798771117237\cdot 10^{-24}\, I$$

```
>evalf(7+1/20*sum(1/k*exp(-6*k)*(2*cos(9*k)-4*sin(9*k)),k=1..5)-1/20*sum(1/k*exp(-20*k)*(2*cos(37*k)-
4*sin(37*k)),k=1..5),24);
```

6.99957050820771214882333

The above answer obtained by Maple appears $I\ (=\sqrt{-1})$, it is because Maple calculates by using special functions built in. The imaginary part is very small, so can be ignored.

Example 2. Using Theorem 2, we can evaluate the following indefine integral

$$\int \frac{e^{-x}\sin x-e^{-5x}\sin 5x+e^{-6x}\sin 4x}{1-2e^{-x}\cos x+e^{-2x}}dx = \frac{1}{2}\cdot\sum_{k=1}^{4}\frac{1}{k}e^{-kx}(-\sin kx-\cos kx)+C \tag{12}$$

for all $x \neq 0$.

Hence, we obtain the definite integral

$$\int_{4}^{9} \frac{e^{-x}\sin x-e^{-5x}\sin 5x+e^{-6x}\sin 4x}{1-2e^{-x}\cos x+e^{-2x}}dx = \frac{1}{2}\cdot\sum_{k=1}^{4}\frac{1}{k}e^{-9k}(-\sin 9k-\cos 9k)-\frac{1}{2}\cdot\sum_{k=1}^{4}\frac{1}{k}e^{-4k}(-\sin 4k-\cos 4k) \tag{13}$$

We also use Maple to verify the correctness of (13).

```
>evalf(int((exp(-x)*sin(x)-exp(-5*x)*sin(5*x)+exp(-6*x)*sin(4*x))/(1-2*exp(-x)*cos(x)+exp(-2*x)),x=4..9)
,14);
```

$$-0.012814751181223 + 4.6888314544889 \cdot 10^{-16}\, I$$

```
>evalf(1/2*sum(1/k*exp(-9*k)*(-sin(9*k)-cos(9*k)),k=1..4)-1/2*sum(1/k*exp(-4*k)*(-sin(4*k)-cos(4*k)),
k=1..4),14);
```

$$-0.012814751181222$$

The above answer obtained by Maple also appears I , the imaginary part is very small, so can be ignored.

IV. Conclusions

As mentioned, Euler's formula, DeMoivre's formula, and finite geometric series play important roles in the theoretical inferences of this study. In fact, the applications of these formulas are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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