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Uniform Cubic Spline Interpolation with Some Additional Techniques for Functions with Strong Gradients

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Abstract:
Spline interpolation is commonly applied in many applications because several desirable interpolation properties are evident and related to the smoothness and accuracy. It may distort somewhat because of the Gibbs phenomenon though it is less noticeable if there is a gap or if the change from one slope to another is steep. Moreover, the issues of data consistency are needed in some applications, but these peculiarities are not checked automatically by interpolation. Thus, it becomes imperative to apply a few more additional technologies to facilitate the uniformity. Orden and repetitiveness of C^2 are not ensured simultaneously, and thus, the last interpolation is not a spline in actuality. The analysis of the given problem is carried out in the context of developing a sufficient condition for generating a homogeneous cubic spline based on the cubic Hermite interpolator and the identification of two construction methods based on nonlinear formulas. As such, these methods are aimed at reducing the regions where accuracy of order is not maximum or where interpolation is less than C^2 order. We compare the order of accuracy and the intrapolar pattern of the extended and the original method and perform many numbers experiment to check with theory.
Keywords: uniformity, cubic Hermite internal polarities, cubic spline internal polarities, non-linear calculation of derivatives

I. INTRODUCTION

The strengths of interpolation, of which Lagrange interpolation is one, are that the constructed interpolator is expressed by a simple equation, so once getting y_0, y_1, y_0, y_1, y_n, they can be easily obtained as these are real life phenomenon and can be experienced by anyone. It is needed to mention that some numerical applications do not need as sharp interpolation as in the case of Lagrange polynomials. Graphics pioneers employed thin flexible strip known as splines which are employed to plot curves through a set of points. Likewise, you can also ‘get least squares fit’ of a smooth curve through a set of points (x_i, y_i) without knowing first derivatives (which are often unseen entirely). The issue arises from the fact that the grid of the set Δ is t_0, t_1, t_n in the interval [a, b]and the set of the corresponding y values $y_0, y_1, \ldots y_n$. the widths of the partitions $\{t_0, t_1, t_n\}$ and the function $s \in C^1$ [a, b] where $s(t_i)=y_i$ for $i=0,1,\ldots,n$. [1].
The spline has multiple applications, but in mathematics, it is extremely crucial because a cubic spline is usually incorporated. To define a uniform tertiary spline, it is necessary to use a grid $\Delta= \{t_0, t_1, \ldots, t_n=b\}$ is given

on the interval [a, b]. The function s: Suppose $y_0, y_1, \in R$ then $[a, b] \rightarrow R$ is the uniform cubic spline interpolating the data., y_n on the grid Δ provided that: y_n on the grid Δ provided that:

$s(t_i) = y_i, i=0, 1, n$

$S \epsilon M_2^3 (\Delta)$

If s interpolates the data $\{ f(t_0), f(t_1), f(t_n)\}$ for f: Given that f: $[a, b] \rightarrow R$, we can state that s is the function of Interpolates on Δ. [2]

The aim of this study is to interpolate homogeneous cubic splines using some additional methods for functions with strong gradients, which are taken from the main sources [3]. Therefore, to obtain a function with a strong gradient, it is sufficient to use the inverse spline interpolation of Hermitian polynomials. In particular, Hermitian interpolation is used to obtain C^1 class interpolation for numerical solutions of differential equations. [4] Therefore, in this thesis, the problem of Hermitian cubic interpolation in the interval $[x_i, x_{(i+1)}]$ should be considered as follows:

$P(x_i) = f_i, P(x_{(i+1)}) = f_{(i+1)}$

$P'(x_i) = f_i, P'(x_{(i+1)}) = \dot{f}_{(i+1)}$

where $h_i = x_{(i+1)} - x_i$ will be the distances for $\hat{h} = \max_{(i=1, n-1)} (h_i)$. Therefore, we will have as follows:

$P_i(x) = c_1^i + c_2^i (x-x_i) + c_3^i (x-x_i)^2 + c_4^i (x-x_i)^2 (x_i - x_{(i+1)})$

where in

$c_1^i := f_i, c_2^i := f[x_i\, x_i] = \dot{f}_i, c_3^i := f[x_i\, x_i, x_{(i+1)}] = (m_i - \dot{f}_i)/h_i$

$_4^i := f[x_i\, x_i, x_{(i+1)}, x_{(i+1)}] = (\dot{f}_{(i+1)} + \dot{f}_i - 2mi)/h_i^2$

Thus, it is possible to state that applying Hermitian cubic interpolation as some additional method to the uniform cubic spline interpolation can be rather effective and something new for the functions which have strong gradients. Therefore, in this paper the application, analysis and assessment of this method is looked at and with the help of numerical outcomes coupled with the modelling aspect done in MATLAB software it is possible to compare the efficiency of this method to other such methods.

II. THE NECESSITY AND IMPORTANCE OF RESEARCH

Today, spline approximation functions are applied in numerous projects that may be graphology or graphic designs, among others. For example, if you want to write the letter 'L' with a computer using software then it is preferred that you use a spline because here the curve is smooth and approximation function is closer to f function. Also, application of the theory of spline functions is much of concern the solution of boundary value problems for ordinary partial differential equations. The relevance of this topic rises even higher because it has been found out that there is significant research on the subject of homogeneous cubic spline. Non-linear interpolation is the correction of a sequence of data points; ARANDIGA in [5] proposed a new approach by applying non-linear interpolation techniques that provided a good approximation of images. It is in [6] that Cripps tackled on the use of cubic interpolation curve that is uniform in the manipulation of cubic spline functions. In [7], VOLBERG discussed the issues concerning the application of spline functions addressing the matter through optimization. The relevance of this work is also in the lack of inverse of spline interpolation of Hermitian polynomials for functions with strong gradients thus the need to examine on homogeneous cubic splines and use numerical analysis which demonstrates the efficiency of this approach with other similar approaches.

III. RESEARCH BACKGROUND

The main motivation of the study was derived based on from the bibliography indications mentioned in [3]. According to the study, many applications are solved through spline interpolation in view of properties of interpolation that is desirable concerning smoothness and accuracy. However, if there is some sort of break or discontinuity in the data, Gibbs phenomenon may exhibit some sorts of defects. Also, there are applications where the data shall remain exact, and this feature is not realized by interpolation as a matter of course. Hence, several more technologies are to be added to regulate the processes and make them uniform. Oct. This is why the regularity and, in particular, the from C^2 are not given at the same time, and as a result, the obtained interpolation can hardly be referred to as a curve. A necessary and sufficient condition for constructing a homogeneous cubic spline on the basis of the Hermitian cubic interpolant has been investigated and two methods of constructing cubic splines by means of nonlinear equations have been described. These methods are created to eliminate areas where the order of precision is not maximum or where interpolation is of the order of C^2. In both cases, several numerical studies were made to check the efficiency and order of the approximations of intrapolar patterns as well as to compare the obtained numerical results with the theoretical ones.

Another reason for this investigation was introduced by the reference [8] that suggested the new colour correction for images in which the scheme of the colour map transfer is based on a uniform HERMITIAN tertiary spline, and it propagates slowly through the target image. Source analysis is done as well as target analysis. Consequently, curved models may work with nonlinear colour maps, whereas the idea of reducing the gradients' variation decreases dramatic changes in the image structure. Adaptive heuristics have been used to keep the effective searches within a lower Decrement space and thus, the computation time shortened. The effectiveness of the proposed method is supported by experimental evaluations based on two settings of state-of-the-art linear mapping models.

Another motivation is borrowed from the reference [9] elucidating the application of the tertiary splines for the interpolation of homogeneous data sets. Interpolated 3. Here, the order-order splines are preferred for data fitting, which are defined with the help of low-order polynomials, have the continuity of c^2, and are capable of meeting the necessary smoothness criteria. Unfortunately, the same restrictions often time do not hold another desirable property – uniformity. A set of data points that increase (or decrease) uniformly can create irregularity in the curve; that is, curve oscillations. In such cases, is loses some of its softness, but this has to be done to ensure homogeneity of the material.

IV. Research Objectives

Study and investigation of uniform cubic spline interpolation with some additional techniques for functions with strong gradient.

V. Research Questions

Will uniform cubic spline interpolation lead to stable convergence of functions with strong gradients?
Will monotonic interpolation by cubic spline C^2 lead to solutions for functions with strong gradients?

VI. Research Assumptions

The interpolation of the uniform cubic spline will lead to the convergence of stability of functions with a strong gradient.
Monotonic interpolation by cubic spline C^2 will lead to solutions for functions with strong gradients.

VII. Spline

Cartographers have long used long and narrow strips of flexible material, somewhat similar to French curves, to represent smooth curves passing through a set of assumed points, these strips are called splines. A smooth curve can be drawn through these points by holding the spline at various suitable points and changing the position of the spline using weights called "marks".

Thus, a spline is defined as a flexible strip that can be held with a weight "label" so that it passes through any point, but the modern mathematical method of smooth transition to the law of curvature is an adaptation of this idea. The definition of a spline was first introduced in 1946 in the famous Schoenberg paper.

Naturally, the starting point for studying the spline function is the tertiary spline. You can see that this spline is an effective tool for approximation and interpolation. A cubic spline is usually defined by a cubic set (cubic polynomial) at a point (label) and replaces the cartographer's spline with a new polynomial at each interval. Higher degree polynomials are actually undesirable for small intervals. Because complex functional relationships are usually undesirable.

To match the cartographer's view of the spline, both the slope (dy⩽dx) and the curvature ((dy^2)⩽(dx^2)) are the same in two 3rd-order curves connected at some point.
The third polynomial "1" and its first and second derivatives are continuous.

VIII. The basis of the theory

Denote an arbitrary selection from the interval [a, b] as Δ = {a = x_0 < x_1 < ⋯ < x_n = b} Definition 20. A cubic spline function S_Δ on Δ is a real function S_Δ: Define a function [a, b] →R with the following properties

1. We have S_Δ ∈ C^2 [a, b], which means for the first partial derivatives of S_Δ, there is continuity on the interval [a, b].
2. The approximations S_Δ on the subinterval [x_i, x_(i+1)] for i=0,1 are to be found. which, denoting the element in the |i|th position as ai, with i ranging from 0 to n-1, relates to a polynomial of order not more
3. Thus, a cubic spline means a set of cubic polynomials that are connected continuously and the values and first and second-order derivatives of which are equal at the node points x_i for i=1., n-1. I defined the finite sequence Y as Y= (y_0, y_1, … y_n) n+1 real numbers known as the knots of the sequence.

Definition 21. To indicate a spline function in the finder S_Δ, we write S_Δ (Y;) Such that S_Δ (Y;) = Y_i for i=0, 1., we have: S_Δ (Y; x_i) =y_i holds true and there are three mandatory conditions which are as follows:

1.$S''_{\Delta}(Y;a) = S''_{\Delta}(Y;b) = 0$

2. $k = 0,1,2,S_{\Delta}^{(k)}(Y;a) = S_{\Delta}^{(k)}(Y;b)$ And $S_{\Delta}(Y;.)$,is intermittent numbers for

3. y'_0 and y'_n :we have $S'_\Delta(Y;a) = y'_0$ and $S'_\Delta(Y;b) = y'_n$

it shows Let each of these conditions alone show the uniqueness of the interpolating spline function to give One The necessary condition for condition 2 of (21) is that for this purpose and to prove the minimum property spline function, per The set of all real functions belonging to the set $L^2[a,b]$ So that $f^{(m-1)}$ It is $K^m[a,b]$ absolutely $[a,\mathrm{b}]$ continuous to give $K^m[a,b]$ to $f^{(k)}(a) = f^{(k)}(b)$ for each $k = 0,1 \ldots, m-1$ With $K_p^m\,[a,b]$ it shows to give these functions are intermittent, with periodicity we say pay attention Let's say that if condition 2 of (21) is satisfied, \ then $S_\Delta \in K_p^3$ And If , And $S_\Delta(Y,.) \in K_p^3$ If $f \in k^2[a,b]$ then we define:

$$\|f\| = \sqrt{\int_a^b |f''(x)|^2 dx} \tag{1.2}$$

Pay attention let's do that. Such One false soft Is Because Condition if then it may not be valid. For example, for the function that and are fixed is but it is not equal to zero. Theorem 22. We will take and an exception from the interval to be If is a spline function with nodes, then:

$$\|f - S_\Delta\|^2 = \|f\|^2 - \|S_\Delta\|^2 -$$
$$-2\left[\left(f'(x) - S'_\Delta(x)\right)S''_\Delta(x)\,\Big|_a^b - \sum_{i=1}^{n}\left(f'(x) - S'_\Delta(x)\right)S'''_\Delta(x)\,\Big|_{x_{i-1}^+}^{x_i^-}\right].$$

(Note Let it be a piecewise constant function with discontinuity on the nodes Is. Therefore, I must use the left and right boundaries of and that with and it shows to give) proof According to the definition we have:

$$\|f - S_\Delta\|^2 = \int_a^b \|f''(x) - S''_\Delta(x)\|^2\, dx$$

$$\|f\|^2 - 2\int_a^b f''(x)S''_\Delta(x)dx + \|S_\Delta(x)\|^2$$

$$\|f\|^2 - 2\int_a^b \left(f''(x) - S''_\Delta(x)\right)S''_\Delta(x)dx - \|S_\Delta(x)\|^2.$$

By partial integration, we have $i = 1,2, \ldots n$

$$\int_{x_{i-1}}^{x_i} \left(f''(x) - S''_\Delta(x)\right)S''_\Delta(x)dx.\left(f'(x) - S'_\Delta(x)\right)S''_\Delta(x)\,\Big|_{x_{i-1}}^{x^i}$$

$$-\int_{x_{i-1}}^{x_i} \left(f''(x) - S''_\Delta(x)\right)S'''_\Delta(x)dx.$$

$$\left(f'(x) - S'_\Delta(x)\right)S''_\Delta(x)\,\Big|_{x_{i-1}}^{x^i} - \left(fx - S_\Delta(x)\right)S'''_\Delta(x)\,\Big|_{x_{i-1}^+}^{x_i^-}$$

$$+\int_{x_{i-1}}^{x_i} \left(fx - S_\Delta(x)\right)S_\Delta^{(4)}(x)dx,$$

Because $S_\Delta^{(4)} \equiv 0$ on the subintervals (x_{i-1}, x_i) and f and $S''_\Delta(x)$ the continuum on $[a,b]$ are.

We also have:

(2.2) $$\sum\left(f'(x) - S'_\Delta(x)\right)S''_\Delta(x)\Big|_{x_{i-1}}^{x^i} = \left(f'(x) - S'_\Delta(x)\right)S''_\Delta(x)\Big|_a^b.$$

And the proof of the theorem is complete to be with the help of this theorem, the important property of soft minimum of spline functions is proven do.

$\Delta= \{a = x_0 < x_1 < \cdots < x_n = b\}$ Arbitrary output from the interval $[a, \mathrm{b}]$ Amounts $Y = (y_0, \ldots, y_n)$ and function $f \in K^2(a, b)$ with $f(x_i) = y_i$ for $i = 0,1 \ldots, n$ then $\|f\| \geq \|S_\Delta(Y; .)\|$ And more precisely the following relationship:

$$\|f - S_\Delta(Y; .)\|^2 = \|f\|^2 - \|f - S_\Delta(Y; .)\|^2 \geq 0,$$

For any spline function $S_\Delta(Y; .)$ provided that one of the following conditions is met:

1.$S''_\Delta(Y; a) = S''_\Delta(Y; b) = 0$

2.$f \in K_p^2[a, b]$ and$S_\Delta\,(Y; .)$

3.$S'_\Delta(Y; a) = f'(a)$And$S'_\Delta(Y; b) = f'(b)$

In either case spline function$S\,(Y; .)$.It is uniquely determined Proof If $S_\Delta\,(Y; .) = S_\Delta$ In each of the above three conditions, the following statement applies : In either case spline function $S\,(Y; .)$ It is uniquely determined proof If $S_\Delta\,(Y; .) = S_\Delta$ In each of the above three conditions, the following statement applies :

$$\left(f'(x) - S'_\Delta(x)\right)S''_\Delta(x)\Big|_a^b - \sum_{i=1}^{n}\left(f'(x) - S'_\Delta(x)\right)S'''_\Delta(x)\Big|_{x_{i-1}^+}^{x_i^-} = 0,$$

According to theorem (22), it is zero to be This property proves the minimum of the spline function slow down $\bar{S}_\Delta\,(Y; .)$ To prove its uniqueness, it is assumed do Another spline function with the same property Hi $S_\Delta\,(Y; .)$ be Suppose $S_\Delta\,(Y; .)$ the function role plays $f \in K^2(a, b)$ in the case. A feature of Minimum $S_\Delta\,(Y; .)$ we have: to fulfill the case. We have minimum features: $S_\Delta\,(Y; .)$

$$\left\|\bar{S}_\Delta\,(Y; .) - S_\Delta(Y; .)\right\|^2 = \|\bar{S}_\Delta(Y; .)\|^2 - \|f - S_\Delta(Y; .)\|^2 \geq 0$$

and because$S_\Delta(Y; .)$ and$\bar{S}_\Delta(Y; .)$ may can play any role, we have$f$:

$$\left\|\bar{S}_\Delta\,(Y; .) - S_\Delta(Y; .)\right\|^2 = \int_a^b |\bar{S}''{}_\Delta(Y; .) - S''_\Delta(Y; .)|^2 dx = 0.$$

Because $S''_\Delta(Y; .)$ and $\bar{S}''{}_\Delta(Y; .)$ both are continuous, we have: As a result, by integrating the sides, we have:

$\bar{S}_\Delta\,(Y; x) \equiv S_\Delta(Y; x) + cx + d$

But for $x = a$ And$x = b$:we have$\bar{S}_\Delta\,(Y; x) = S_\Delta\,(Y; x)$this is the result give that And$c = d = 0$

The feature of the soft minimum of the spline function expressed in theorem (23) in the case that condition 1 of (21) If it is established, it will be the result gives that among all the functions belonging $K^2[a, b]$to$f(x_i) = y_i$

With exactly the function $S_\Delta\ (Y;x)$ with condition $S''_\Delta(Y;x) = 0$ in points $x = a, b$ may Let minimize the following integral slow

$$\|f^2\| = \int_a^b |f''(x)|^2 dx$$

Therefore, the spline function in mode 1 of (21) refers to the normal spline function, and in mode 2 and 3 spline functions $S_\Delta(Y;.)$ in order on the set More limited $K_p^2[a,b]$

$$\{K_p^2[a,b] \big| f'\ (a) = y'_0, f'_b = y'(n)\} \cap \{f \mid f(x_i) = y_i for\ i = 0,1,n\}$$

minimum do Phrase $f''(x)(1 + f'(x)^2)^{-3/2}$ The degree of curvature of the function f in $x \in [a,b]$ in shows to give If $|f'(x)|$ is smaller compared to 1, then the amount of curvature is almost equal $f''(x)$ to Is. the amount of $\|f\|$ It gives us an approximate measure of the curvature of the function in the interval $[\mathrm{a}, \mathrm{b}]$ give That is, the normal spline function is a smooth function whose points (x_i, y_i) in exchange for $i = 0,1, \ldots n$ May interpolate slow

IX. NUMERICAL RESULTS

In this section, we show experiments to confirm the theoretical results obtained so far. In particular, we divide the experiment into 2 parts:

According to the order of 4.1, use a smooth function or a piecemeal smooth function to examine the accuracy of different reconstructions. We are conducting two experiments on both evenly spaced and non-spaced grids. In the first case, method B and the MOON are the same.

in 4.2, we are looking at uniformity when the function is unknown and only node values are given.

The calculated values that do not comply with the conditions of theorem 1.4 are replaced by new approaches using the described method.

R_k: The described method has the boundary conditions S^'(a) = f^'(a), S^'(b) = f^'(b) 3. it is a line of order, but instead uses the system (4.1)

$$A_{i_0} = \begin{bmatrix} A_{i_0^-} & \\ & A_{i_0^+} \end{bmatrix}, \dot{F}_{i_0} = \begin{bmatrix} \dot{F}_{i_0^-} \\ \dot{F}_{i_0^+} \end{bmatrix}, B_{i_0} = \begin{bmatrix} B_{i_0^-} \\ B_{i_0^+} \end{bmatrix}$$

It is made because any approximate value of the derivative is changed with the change in the step size and consequently with the displacement of the abstract point itself. In this case, for methods O and R the symbol k will be adduced to show what sort of derivative approximation is used; hence k=FB, B, AY or Eqs.

$$\dot{f}_i^{FB} := \begin{cases} \dfrac{3m_{i-1}m_i}{m_{i-1} + 2m_i}, & \text{if} \quad |m_i| \le |m_{i-1}|; \\ \dfrac{3m_{i-1}m_i}{m_i + 2m_{i-1}}, & \text{if} \quad |m_i| > |m_{i-1}| \\ 0, & \text{if} \quad m_{i-1}m_i \le 0. \end{cases}$$

$$\dot{f}_i^{B} := \begin{cases} \dfrac{(wl_i + wr_i)m_{i-1}m_i}{wl_i m_i + wr_i m_{i-1}}, & m_i m_{i-1} \ge 0 \\ 0 & m_i m_{i-1} < 0 \end{cases}$$

&

$$\dot{f}_i^{AY} := \begin{cases} \operatorname{sign}(m_i) \dfrac{\left(h_i + h_{i-1}\right)^{1/p_i} \left|m_{i-1}\right| |m_i|}{\left(h_{i-1}\left|m_{i-1}\right|^{p_i} + h_i |m_i|^{p_i}\right)^{1/p_i}} & \text{if} \quad m_i m_{i-1} \geq 0; \\ 0 & \text{if} \quad m_i m_{i-1} < 0, \end{cases}$$

Table 4.1.

(grid with equal spacing) and estimated accuracy order Experiment 1 with h □ $= 2^{-l}$

$. W_1 = \left\{i: 2 \leq i \leq 2^{l+1} - 1\right\}$، $\log_2\left(e_l^{W_1} / e_{l-1}^{W_1}\right), 5 \leq l \leq 8$

h	S	$\mathbf{R}_{FB}$	$\mathbf{O}_{FB}$	$\mathbf{R}_B = \mathbf{R}_{AY}$	$\mathbf{O}_B = \mathbf{O}_{AY}$
$3.125e{-2}$	3.9988	0.9390	0.9390	1.9952	1.9952
$1.562e{-2}$	3.9997	0.9715	0.9715	1.9988	1.9988
$7.812e{-3}$	3.9999	0.9863	0.9863	1.9997	1.9997
$3.906e{-3}$	3.9999	0.9932	0.9932	1.9999	1.9999

Table 4.2.

Experimental part 1, when h=2^(-l) (the step of the grid is equal) estimated accuracy of the order $\log_2\left(e_l^{W_2} / e_{l-1}^{W_2}\right), 5 \leq l \leq 8$ ، $W_2 = W_1 \setminus \left\{i_0\right\}$.

h	S	$\mathbf{R}_{FB}$	$\mathbf{O}_{FB}$	$\mathbf{R}_B = \mathbf{R}_{AY}$	$\mathbf{O}_B = \mathbf{O}_{AY}$
$3.125e{-2}$	3.9988	0.9390	3.9988	1.9952	3.9988
$1.562e{-2}$	3.9997	0.9715	3.9997	1.9988	3.9997
$7.812e{-3}$	3.9999	0.9863	3.9999	1.9997	3.9999
$3.906e{-3}$	3.9999	0.9932	3.9999	1.9999	3.9999

precision

Divide this section into 2 sections: Divide this section into 2 sections:

First, the situation of parallel array is analyzed. Make sure that order of the accuracies of the approximaty to the value of the derivation of smooth part is of the order O (h 4).

This was followed by an experiment of realizing a function as a sum of vectors by using inhomogeneous lattice. In both cases, perform the check taking into consideration the points in reference to the distance to the discontinuity point.

It is only advisable to test on a uniform network, and intuitively, it was expected that the new link would affect all the participating networks.

Here in this section, we will discuss 2 tests. :

In the first experiment the function is smooth as in the previous experiment and replaces the differential approximation at some point and sees how this new value influences the smoothness and accuracy of the curve on the plot.

As for the second example, it is noted that piecewise smoothing functions with jump discontinuities are considered.

Experiment 1. To check the precision order of the methods, consider the following neat function:In order to evaluate the precision order of the methods, it is advisable to examine the performance of the neat function defined by: Referring to the above-mentioned problem, the general functions can be described as: f (x) = x ^ 4 + sin θ (x)

To it we add the smooth grid x_j^l = j / 2^l , where j = 0, 1, …, 2^l – 1. . . , 2 ^ (l + 1): limits to use in separating is [0, 2]. l is intended to be the observation of the number of previous similar events and as such, is described to be a 'non-negative integer', meaning it should be 0 or greater but also strictly greater than 0. In accordance with the type of the numerical experiment, only some number of sampling points is selected for the accurate comparison of various methods. w ⩽ 0 . . . create 2 ^ (l + 1). Therefore, if you want to see the impact that has been provided by Dec, you will have to initiate the window with the Dec On the other hand, in the previous

Section, it was established that there is a Max value of the order of accuracy a distance away from the gap Therefore, to verify this, one needs choosing window that doesn't contains the boundaries of the gap accordes to the result of (2). 10. portion 2. 11. The indicated order of errors and the approximation of various methods are determined at the selected point. An error occurred while using:

$$e_l^W = \max_{i \in W} \left| f'(x_i) - \dot{f}_i \right|$$

And the order of accuracy with calculations is estimated.

$$o_l^W = \log_2 \left(\frac{e_l^W}{e_{l-1}^W} \right)$$

If the error expression is greater than the Taylor expansion of the method's precision order, then the o_l^W would approach the value of the precision order in the limit of the network size.

Therefore, in Method 0 windows the formal value of the derivative f_ (2 ^ l), which before was 2 ^ l = R i_0, replaceable by value computers using the method of section 3 to affirm the precision and smoothness properties elaborated in sections 2. 2 and 2. 3.

First, consider the window W_1 ={I: where, if l is int then I = 0 , 1 , 2 , …. , l or if l is non integer then i = 0, 1, 2, …. {l + 1}. In these settings, orders of check is evaluated by putting the first derivative made at the i_0 point around. The second of accuracy of the quadratic term yielded according to the relation Table 4 to B and AY is 1, is:

$$\dot{f}_i^{AY} := \begin{cases} \operatorname{sign}(m_i) \dfrac{\left(h_i + h_{i-1}\right)^{1/p_i} \left|m_{i-1}\right| \left|m_i\right|}{\left(h_{i-1}\left|m_{i-1}\right|^{p_i} + h_i \left|m_i\right|^{p_i}\right)^{1/p_i}} & \text{if} \quad m_i m_{i-1} \geq 0; \\ 0 & \text{if} \quad m_i m_{i-1} < 0, \end{cases}$$

we get For the FB method; it is decreased by the equation f ̅_i^fb=f'(x _i)+O(h _).

Finally, it should be noted that together with ∥F ̇-F^'∥_∞⩽O(h^4), the order of the algorithm is four S due to the uniformity of the network. However, if the window, in terms of which the differences are calculated, does not contain i_0 for W_2 = W_1 \ {i_0 }, then the order of accuracy of the methods rises up to 4rd order, in grounds of Proposition 2. 6. On the other hand, as Corollary 2 shows, these penalties are higher in regard with the R methods. 10 which proved that the order of the accuracy does not increase. (see Table 4.2).

Table 4.3.

Experiment 1 with □ $= 2^{-l}$ (grid with equal spacing) and estimated accuracy orders $\log_2 \left(e_l^{W_3} / e_{l-1}^{W_3} \right), 5 \leq l \leq 8$ ، $W_3 = \left\{ i: 1 \leq i \leq l_0 \text{ بـ } l_1 \leq i \leq 2^{l+1} - 1 \right\}$

h	S	$\mathbf{R}_{FB}$	$\mathbf{O}_{FB}$	$\mathbf{R}_B = \mathbf{R}_{AY}$	$\mathbf{O}_B = \mathbf{O}_{AY}$
$3.125e{-2}$	3.9988	2.8397	3.9988	3.8882	3.9988
$1.562e{-2}$	3.9997	2.8713	3.9997	3.9067	3.9997
$7.812e{-3}$	3.9999	2.8864	3.9999	3.8930	3.9999
$3.906e{-3}$	3.9999	2.8932	3.9999	3.9063	3.9999

Table 4.4.

The second experiment was carried out using h=2^(-l) orders of accuracy where l is the number of bits used to estimate log_2($\log_2\left(e_1^{\tilde{W}_3}/e_{l-1}^{\tilde{W}_3}\right), 5 \le l \le 8, \quad W_3 = \left\{i: 1 \le i \le l_0 \text{ بي } l_1 \le i \le 2^{l+1} - 1\right\}$. ، when the network is uniform.

h	**S**	$\mathbf{R}_{FB}$	$\mathbf{O}_{FB}$	$\mathbf{R}_B = \mathbf{R}_{AY}$	$\mathbf{O}_B = \mathbf{O}_{AY}$
$3.125e{-}2$	0.8964	1.8706	0.8964	2.0121	0.8964
$1.562e{-}2$	0.8982	1.7824	0.8982	1.7898	0.8982
$7.812e{-}3$	0.8991	1.8395	0.8991	1.8448	0.8991
$3.906e{-}3$	0.8995	1.8693	0.8995	1.8723	0.8995

LASTLY, IF SETTINGS PERTAINING TO PROPOSITIONS 2. 8 AND 2. 9 WITH

- (&L_0=(I_0-1)+LOG_2(H ̂)=(2^L-1)+LOG_2(2^(-L))=2^L-L-1,@&L_1=(I_0+1) -LOG_2(H ̂)=(2^L+1)-LOG_2(2^(-L))=2^L+L+1,)

  • DEFINITION W_3 = {I:1 ⩽ I ⩽ L_0 : L_1 ⩽ I ⩽ 2 ^(L + 1) - 1} SIMILARLY THE OPTIMUM VALUES OF I IS STATED AS FOLLOWS IN TABLE 4. 3, THE ACCURACY R METHOD AND THE B = MONTH METHOD ARE A METHOD FOR R_FB. IN THIS CASE, THE SIZE OF THE WINDOW CHOSEN IN RETEROSPECT TO THE DISCONTINUITY IS MORE THAN SUFFICIENT TO PROVE THAT THE ORDER OF ACCURACY GIVEN IN TABLE 4 IS ENHANCED. 2 ON THE REMAINING POINTS. BUT IF WE SIMPLY REALISE THAT DISTANCE CAN BE REDUCED IF LIMITS STATED IN THE FORMULA ARE CONSIDERED. THESE ARE THE FORMULAE USED FOR EQUALLY SPACED GRIDS; (30) HENCE, THE GENERATED CURVE FOR FACILITY LOCATION HAS THE MAXIMUM PRECISION AND ORDER IN THE DECEMBER X_1 TO X_ OF ALL THE METHODS DERIVED FROM LEMMA 1. 5 .

  SECOND EXPERIMENT. WE ARE DEALING WITH FUNCTIONS THROUGHOUT JOINED WITH JUMPS WHICH OCCUR IN DECEMBER [X_ (I_0), X_(I_0 +1)]. THE PRESENCE OF DISCONTINUITIES CREATES 2 EFFECTS:THE PRESENCE OF DISCONTINUITIES CREATES 2 EFFECTS:

  FIRST, IN ACCORDANCE WITH THE EQUATION (4. 1), THE DERIVATIVE APPROXIMATION SUFFERS FROM THE GIBBS PHENOMENON AND GENERATES ARTIFICIAL RATHER THAN PHYSICAL OSCILLATIONS NEAR DISCONTINUITIES; THUS, MONOTONICITY IS NOT PRESERVED.

  SECOND, THE METHOD DESCRIBED IN SECTION 3 WITH RESPECT TO LDL! C AND HDL! C DOES NOT REACH ITS MAXIMUM ACCURACY.

FUNCTION

$$g(x) = \begin{cases} x^4 + \sin(x) & 0 \le x \le 1 \\ 4 + x^4 + \cos(x), & 1 < x \le 2 \end{cases}$$

This we address and confine to the range [0,2] as done in the previous Subsection. This function has jump discontinuity at x = 1 which is in agreement with nodes x (2^l). Actually, in context of comparing the order for the convergence of the difference equation, one alters only the value of the first order derivatives , and thus, it does not meet the criteria of Theorem 1. 4.

In Table 4. 4, from it represent the number orders of the accuracy in the differentiation approximation

$$\tilde{W}_3 = W_3 \setminus \left\{2^l + l + 1\right\}$$

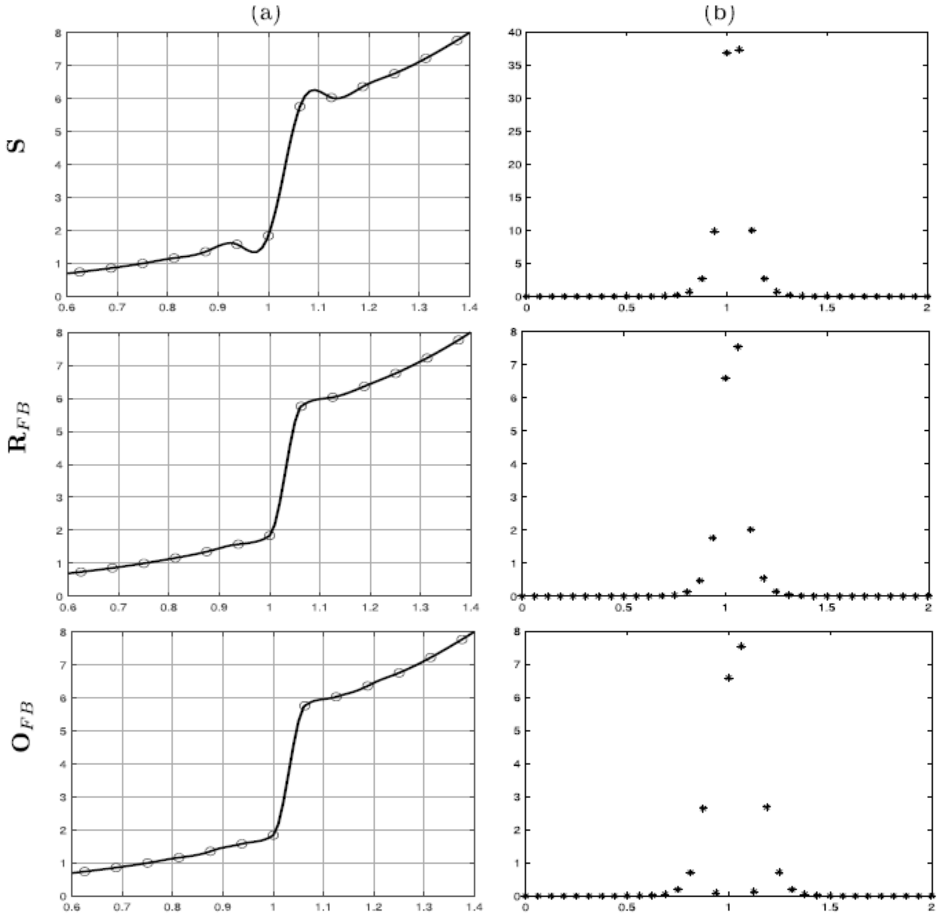
Similarly to W_3 defined in the previous experiment, the left column in Figure 4 is the same. 1 Southern refers to the presentation of the reconstruction based on various methods. The lesson that can be learned from the result of the S-spline is the vibration originating from the lack of the condition of the restriction of uniformity and that the accuracy of reconstruction of the S-spline is low when it comes to orders.

Thus, as in the case of S-spline, the result of O method is O because the first derivative is approximated at the points belonging to W with the accuracy _3. Finally, at the level of using the considered method, the order of accuracy is increased at the last stage. R Next to Figure 4 there is a graph. 1 shows the mistake that was made in the differential calculation.

In Figure 4. 1, using the above-described method of altering approximates to differential value {f_(2^l+ s)} ^2, f (2^l -1) f (2 ^ l + 2)test, it is proved that the theorem 1 is not valid. 1, thereby the spline will be 0 that mean the change in slope is 0 at the given point.

Also, it is necessary to note the scale of the y-axis in Figure 4, Right due to the fact that value differences in different graphs of the same dependency are very large, which is 1. Hence, it can be noted that all methods produce large errors at the breaks, and this is in line with the spline reconstruction and its error is almost one order of magnitude higher than the other methods. B and AY methods do produce fewer errors compared to fb that produces 1/2 errors.

In addition, as you can suppose, seeing that the method r > 0 near [x_(i_0) , x_(i_0 + 1)], the accuracy of method dwindles. In Table 4. While analyzing the data from input id 4, it can be observed that the maximum accuracy has not been obtained as it is desired. According to statement 2. 11, we consider the window l ˜_1⩽i ⩽2^(l+1)-1} or W_4={i:1⩽i⩽l ˜_0 ┤ (&l ˜_0=(i_0-1)+rlog_2(h ̂)=(2^l-1)+rlog_2(2^(-l))=2^l-r·l-1,@&l ˜_1 =(i_0+2)-rlog_2(h ̂)=(2^l+2)-rlog_2(2^(-l))=2^l+r·l+2,)



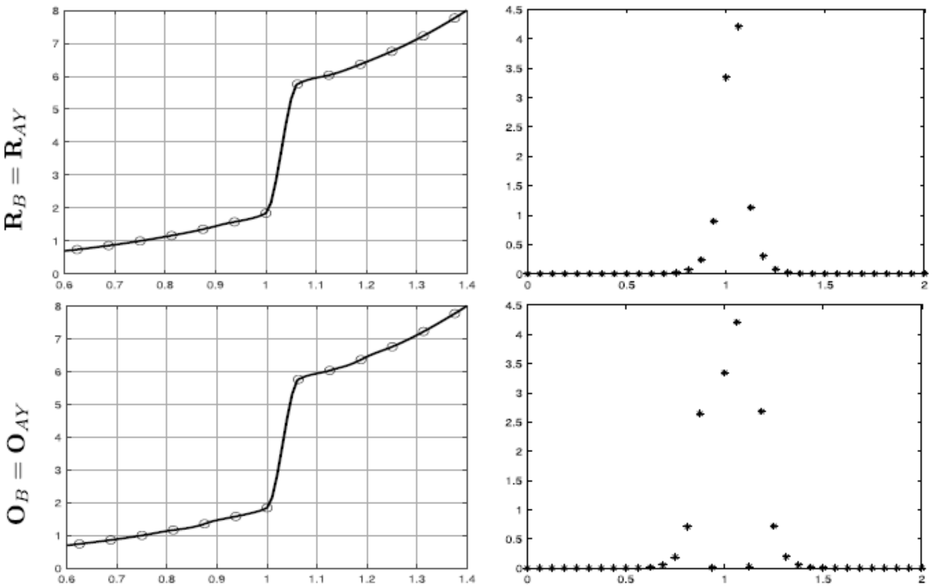


Figure 4. 1. Second.1 Experiment = 4 (a) reconstructs uses different method , b / g ^ (x_i) - g I ^ k /, i = 0,. .. , 32. and r = 2. Table 5 presents the results of this window in the previous section of this paper using the above-mentioned approach. The order of the estimated precision of 0 and S is thus consistent, but the same is not the case for R, for the fact that this is usually arrived at through a derivative gives the method or approach used a bearing on its order of precision.

Table 4.5.

Experiment 2 with $h = 2^{-l}$ (grid with equal spacing) and orders

$$W_4 = \left\{ i: 1 \leq i \leq \tilde{l}_0 \; \_ \; \log_2\left(e_l^{W_4}/e_{l-1}^{W_4}\right), 5 \leq l \leq 8 \; \tilde{l}_1 \leq i \leq 2^{l+1} - 1 \right\}$$

h	S	$\mathbf{R}_{FB}$	$\mathbf{O}_{FB}$	$\mathbf{R}_B = \mathbf{R}_{AY}$	$\mathbf{O}_B = \mathbf{O}_{AY}$
$3.125e{-}2$	2.7961	3.7667	2.7961	3.9049	2.7961
$1.562e{-}2$	2.7980	3.6828	2.7980	3.6906	2.7980
$7.812e{-}3$	2.7990	3.7397	2.7990	3.7452	2.7990
$3.906e{-}3$	2.7995	3.7695	2.7995	3.7727	2.7995

Table 4.6.

Experiment 1 with $\hat{h} = \frac{3}{4}2^{-l}$ (grid with equal intervals) and orders

$$\log_4\left(e_l^{W_1^n}/e_{l-2}^{W_1^n}\right), 3 \leq l \leq 9, W_1^n = \left\{ i: 0 \leq i \leq 2^{l+2} \right\}$$

$\hat{h}$	S	$\mathbf{R}_{FB}$	$\mathbf{O}_{FB}$	$\mathbf{R}_B$	$\mathbf{R}_{AY}$	$\mathbf{O}_B$	$\mathbf{O}_{AY}$
$9.3750e-02$	2.9903	1.1996	1.1996	1.0888	1.9675	1.0888	1.9675
$2.3438e-02$	2.9999	1.0791	1.0791	1.0385	2.0074	1.0385	2.0074
$5.8594e-03$	3.0000	1.0220	1.0220	1.0107	2.0027	1.0107	2.0027
$1.4648e-03$	2.9978	1.0057	1.0057	1.0027	2.0007	1.0027	2.0007

Table (4.7) first Experiment with h ˆ=3/4 2^(-l) (the grids with equal intervals) and orders

Estimate the accuracies log _ 4 $\log_4\left(e_l^{W_2^n}/e_{l-2}^{W_2^n}\right), 3 \leq l \leq 9, W_2^n = W_1^n \setminus \left\{ i_0 \right\}$

Table 4.7.

First Experiment with $\hat{\square} = \frac{3}{4} 2^{-l}$ **(grid with equal intervals) and orders**

Estimated accuracy log_4 $\log_4\left(e_l^{W_2^n} / e_{l-2}^{W_2^n}\right), 3 \le l \le 9, W_2^n = W_1^n \setminus \{i_0\}$

$\hat{h}$	S	R_{FB}	O_{FB}	R_B	R_{AY}	O_B	O_{AY}
$9.3750e-02$	2.9903	1.1623	2.9903	1.0653	1.9190	2.9903	2.9903
$2.3438e-02$	2.9999	1.0760	2.9999	1.0368	1.9964	2.9999	2.9999
$5.8594e-03$	3.0000	1.0218	3.0000	1.0106	2.0000	3.0000	3.0000
$1.4648e-03$	2.9978	1.0056	2.9978	1.0027	2.0000	2.9978	2.9978

Table 4.8.

Experiment 1 with $\hat{\square} = \frac{3}{4} 2^{-l}$ **(grid without equal intervals) and orders**

Estimated accuracy $\log_4\left(e_l^{W_3^n} / e_{l-2}^{W_3^n}\right), 3 \le l \le 9,$ **& log_4**$W_3^n = \{i: 0 \le i \le l_0^n$ ب $l_1^n \le i \le 2^{l+2}\}$

$\hat{h}$	S	R_{FB}	O_{FB}	R_B	R_{AY}	O_B	O_{AY}
$9.3750e-02$	2.9903	2.3242	2.8853	2.0628	2.4435	2.8853	2.8853
$2.3438e-02$	2.9999	2.9997	2.9999	2.9995	2.9998	2.9999	2.9999
$5.8594e-03$	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
$1.4648e-03$	2.9978	2.9978	2.9978	2.9978	2.9978	2.9978	2.9978

Table 4.9.

Second Experiment with $\hat{\square} = \frac{3}{4} 2^{-l}$ **(grid with equal intervals) and orders**

. Estimated accuracy log_4 $\log_4\left(e_l^{\tilde{W}_3^n} / e_{l-2}^{\tilde{W}_3^n}\right), 5 \le l \le 8, \tilde{W}_3^n = W_3^n \setminus \{l_1^n\}$

$\hat{h}$	S	R_{FB}	O_{FB}	R_B	R_{AY}	O_B	O_{AY}
$9.3750e-02$	0.9750	1.4998	0.9750	1.4742	1.5462	0.9750	0.9750
$2.3438e-02$	1.0791	2.0050	1.0791	2.0231	2.0569	1.0791	1.0791
$5.8594e-03$	1.0823	2.0514	1.0823	2.0539	2.0647	1.0823	1.0823
$1.4648e-03$	1.0827	2.0749	1.0827	2.0757	2.0786	1.0827	1.0827

They correspond to the methods at the window point. See Table 4.7. However, you cannot use the r method to improve accuracy because the window size is not large enough.

If the window used to predict the degree of certainty is W_3 ^ n, that is, according to propositions 2.8 and 2.9, even if the estimation method or FB is used at more points around i_0, the degree of certainty of the result is not 2, but 3. the result is the value predicted by 2.10. The reason for this lies regarding the periodicity functions uses in examples or in fact that value l _ 0 ^ n , l _ 1 ^ n are from the formula. (4.2) Very large.

Table 4.10.

Second Experiment with $\hat{\square} = \frac{3}{4}2^{-l}$ **(grid without equal intervals) and orders**

Estimated accuracy log_4

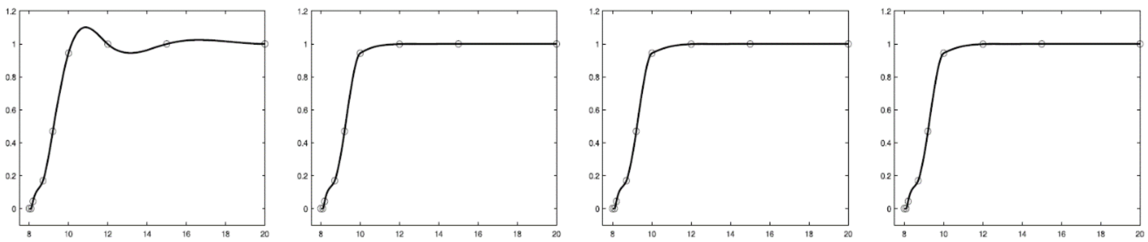
$$.W_4^n = \left\{ i: 0 \le i \le \tilde{l}_0^n \sqcup \tilde{l}_1^n \le i \le 2^{l+2} \right\} \& \log_4\left(e_l^{W_4^n} / e_{l-2}^{W_4^n} \right), 4 \le l \le 10$$

$\hat{h}$	S	R_{FB}	O_{FB}	R_B	R_{AY}	O_B	O_{AY}
$4.6875e-02$	3.1187	3.1856	3.1187	3.1210	3.1391	3.1187	3.1187
$1.1719e-02$	3.1291	3.0872	3.1291	3.0615	3.0616	3.1291	3.1291
$2.9297e-03$	3.1248	3.0209	3.1248	3.0143	3.0139	3.1248	3.1248
$7.3242e-04$	3.1166	2.9547	3.1166	2.9530	2.9528	3.1166	3.1166

Table 4.11.

Test data 3.

i	1	2	3	4	5	6	7	8	9
x_i	7.99	8.09	8.19	8.7	9.2	10	15	12	20
y_i	0	$2.76429e-5$	$4.37498e-2$	0.169183	0.469428	0.943740	0.998636	0.999919	0.999994



S **R_{FB}** **R_B** **R_{AY}**

Figure 4.2. Reconstructions obtained for experiment 3 using different methods.

The algorithm changes the values of nodes i=2,6,7,8. In this case, we obtain uniform reconstruction in the full range (Figure 4.2). The results are very similar in all cases, i.e., both methods produce similar uniform curves because the values that are modified are the most relevant in this example.

MATLAB code:

```
% A program to find the natural cubic spline to fit a set of n=4 x, y
clc, clear all.
x= [1 2 4 5].
y= [0 0 2 2].
n=length(x).
% Need to find A(x), B(x), C(x) and D(x)
% (1) A(x) and C(x)
for j=1: n-1.
 A(j,1:4) = [0 0 -1 x(j+1)]/(x(j+1)-x(j)).
 B(j,1:4) = [0 0 1 -x(j)]/(x(j+1)-x(j)).
 D1= [1].
 C1= [1].
 for i=1:3
 C1=conv (C1, [A(j,3) A(j,4)]).
 D1=conv (D1, [B(j,3) B(j,4)]).
```



```
 end
 C(j,1:4) =(C1-A(j,1:4)) *((x(j+1)-x(j)) ^2)/6.
 D(j,1:4) =(D1-B(j,1:4)) *((x(j+1)-x(j)) ^2)/6.
% Have A(x), B(x), C(x), D(x) need to get yi" y"i+1
AA=zeros (n, n).
BB=(zeros(size(x)))'.
for i=2:(n-1)
 if i-1~=1
 AA (i, i-1) =(x(i)-x(i-1)).
 end
 AA (i, i) =2*(x(i+1)-x(i-1)).
 if i+1 ~=n
 AA (i, i+1) =(x(i+1)-x(i)).
 end
 BB(i)= 6*(((y(i+1)-y(i))/(x(i+1)-x(i))) -((y(i)-y(i-1))/(x(i)-x(i-1)))).
end
% Use MATLABs in built gaussian elimination to solve for unknowns
 ypp=zeros(size(x)).
 ypp (2: n-1) =AA (2: n-1,2: n-1) \BB (2: n-1).
end
 % Add all together to get s(x)
for j=1: n-1
 S (j, :) = A (j, :)*y(j) + B (j, :)*y(j+1) + C (j, :)*ypp(j) + D (j, :)*ypp(j+1).
end
disp ('Each row of the matrix s expresses the 3 polynomials as cubic coefficients').
```

X. Conclusion

An important uniform tertiary internal polarity is determined according to the third line and the nonlinear reconstruction of the derivative at the selected point. In addition to the case when the data have discontinuities or high gradients two modifications which alter the approximation of the derivative at the point which violates the constrain of monotonicity are considered. In the first case you rewrite the spline system and this correct the modified derivative then you use the modified spline system to calculate the derivative on both sides. In this case, the order C ^ 2 remains fixed at all points in the approximation of the subject, except where other approximations in the change of the derivative means that the order of precision reverts to the December in the rest of the section of the subject. This method entails several resolutions of the spline system. 2. Otherwise, the rest of the items of the algorithm remains unchanged and the new approximation of the derivative is used if necessary. Here the points of precision do not interlock or cross, but the order of the differentiation fades away at places where there exists a switch of the derivative. These results are validated by several numerical tests that have been conducted in this study. This work can further be extended in a two-dimensional fashion to get a homogenized bicubic segmented projection.

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MY NAME IS JEIHAN WALEED EBRAHIM AL-AZZAWI, A MASTER'S DEGREE STUDENT IN APPLIED MATHEMATICS WHO, FROM HIS THESIS ENTITLED INTERPOLATION OF A UNIFORM CUBIC SPLINE WITH SOME ADDITIONAL TECHNIQUES FOR FUNCTIONS WITH I HAVE DEFENDED THE STRONG GRADIENT BY OBTAINING THE SCORE OF... AND THE DEGREE OF... I HEREBY UNDERTAKE:

1) THIS DISSERTATION IS THE RESULT OF RESEARCH AND RESEARCH DONE BY ME, AND IN CASES WHERE I HAVE USED THE SCIENTIFIC AND RESEARCH ACHIEVEMENTS OF OTHERS (SUCH AS THESE, BOOKS, ARTICLES, ETC.), IN ACCORDANCE WITH THE EXISTING RULES AND PROCEDURES, THE NAME OF THE SOURCE I HAVE MENTIONED ITS USE AND OTHER SPECIFICATIONS IN THE RELEVANT LIST.

2) THIS THESIS HAS NOT BEEN SUBMITTED BEFORE TO RECEIVE ANY ACADEMIC DEGREE (SAME LEVEL, LOWER OR HIGHER) IN OTHER UNIVERSITIES AND HIGHER EDUCATION INSTITUTIONS.

3) IF, AFTER GRADUATION, I INTEND TO USE AND EXPLOIT THIS THESIS IN ANY WAY, SUCH AS PUBLISHING A BOOK, PATENTING, ETC., I WILL OBTAIN THE RELEVANT PERMITS FROM THE DEPARTMENT OF RESEARCH VICE-CHANCELLOR OF THE UNIT.

4) IF AT ANY POINT OF TIME IT IS PROVEN CONTRARY TO THE ABOVE, I ACCEPT THE CONSEQUENCES AND THE UNIVERSITY UNIT IS ALLOWED TO TREAT ME ACCORDING TO THE RULES AND REGULATIONS AND I WILL NOT HAVE ANY CLAIM IN CASE OF CANCELLATION OF MY ACADEMIC DEGREE.

REFERENCES

[1]. G. Micula, S. Micula, Handbook of Splines, in: Mathematics and its Applications, vol. 462, Kluwer Academic Publishers, Dordrecht, 1999.

[2]. C. Allouch, P. Sablonniere, D. Sbibih, Solving Fredholm integral equations by approximating kernels by spline quasi-interpolants, Numer. Algorithms 56 (3) (2011) 437-453.

[3]. Francesc Aràndiga, Monotone cubic spline interpolation for functions with a strong gradient, Applied Numerical Mathematics 172 (2022) 591–607

[4]. Floater M. S., M.-J. Thus, for Lai, polygonal spline spaces and numerical solution of the Poisson equation is presented in the SIAM J. Numer. Anal. 54 (2) (2016) 797–824.

[5]. F. Aràndiga, A. Baeza & D. F. YŬez, A new class of non-linear monotone Hermite interpolants, Adv. Comput. Math. 39 (2013) 289–309.

[6]. Alternative: The application of C1 monotone cubic Hermite interpolant is discussed by R. J. Cripps and M. Z. Hussain in their paper. Math. Lett. 25 (2012) 1161–1165.

[7]. G. Wolberg, I. Alfy, An energy-minimization framework for monotonic cubic spline interpolation, J. Comput. Appl. Math. 143 (2002) 145–188.

[8]. Fabio Bellavia, Color correction for image stitching by monotone cubic spline interpolation, Pattern Recognition and Image Analysis, 2015, pp 165–172

[9]. George Wolberg, An energy-minimization framework for monotonic cubic spline interpolation, Journal of Computational and Applied Mathematics 143 (2002) 145–188

[10]. A.A. Abushama, B. Bialecki, Modified nodal cubic spline collocation for Poisson’s equation, SIAM J. Numer. Anal. 46 (1) (2008) 397–418.

[11]. H. Akima, A new method of interpolation and smooth curve fitting based on local procedures, J. Assoc. Comput. Mach. 17 (1970) 589–602.

[12]. F. Aràndiga, On the order of nonuniform monotone cubic Hermite interpolation, SIAM J. Numer. Anal. 51 (5) (2013) 2613–2