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RESEARCH ARTICLE

Using Maple to Evaluate the Partial Derivatives of Two-Variables Functions

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Abstract*—This study uses the mathematical software Maple for the auxiliary tool to evaluate the partial derivatives of two types of two-variables functions. We can obtain the infinite series forms of any order partial derivatives of these two types of two-variables functions by using differentiation term by term. At the same time, we provide two examples of two-variables functions to determine their partial derivatives practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

***Keywords*— partial derivatives; two-variables functions; infinite series forms; differentiation term by term; Maple**

I. INTRODUCTION

The computer algebra system (CAS) has been widely employed in mathematical and scientific studies. The rapid computations and the visually appealing graphical interface of the program render creative research possible. Maple possesses significance among mathematical calculation systems and can be considered a leading tool in the CAS field. The superiority of Maple lies in its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. In addition, through the numerical and symbolic computations performed by Maple, the logic of thinking can be converted into a series of instructions. The computation results of Maple can be used to modify our previous thinking directions, thereby forming direct and constructive feedback that can aid in improving understanding of problems and cultivating research interests. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. As for the instructions and operations of Maple, we can refer to [1]-[7].

In calculus and engineering mathematics courses, the study of evaluating the partial derivatives of multivariable functions is an important issue. For example, Laplace equations, wave equations, as well as some other important physical equations are involved the partial derivatives of multivariable functions. Therefore, whether in physics, engineering or other sciences, the evaluation and numerical calculation of the partial derivatives are important and can refer to [8]-[11]. In this paper, we mainly study the evaluation of the partial derivatives of the following two types of two-variables functions

$$f(x,y)=\frac{y^{s}\cos[(n-1)(cx+d)]-r\cos[n(cx+d)]}{y^{2s}-2ry^{s}\cos(cx+d)+r^{2}} \qquad (1)$$

$$g(x,y)=\frac{y^s\sin[(n-1)(cx+d)]-r\sin[n(cx+d)]}{y^{2s}-2ry^s\cos(cx+d)+r^2} \quad (2)$$

, where r,s,c,d are real numbers, n is an integer. We can obtain the infinite series forms of any order partial derivatives of these two types of two-variables functions by using differentiation term by term; these are the major results in this paper (i.e., Theorems 1 and 2), and hence greatly reduce the difficulty of determining higher order partial derivatives values of these two types of two-variables functions. As for the related study of evaluating partial derivatives of two-variables functions can refer to [12]-[16]. On the other hand, we propose two examples of two-variables functions to determine their partial derivatives practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

II. MAIN RESULTS

Firstly, we introduce some notations and formulas used in this study.

Notations.

(i) Let $z=a+ib$ be a complex number, where $i=\sqrt{-1}$, a,b are real numbers. We denote a the real part of z by $\mathrm{Re}(z)$, and b the imaginary part of z by $\mathrm{Im}(z)$.

(ii)Suppose t is any real number, and m is any positive integer. Define $(t)_m=t(t-1)\cdots(t-m+1)$, and $(t)_0=1$.

(iii)Assume q,p are non-negative integers. For the two-variables function $f(x,y)$, the p -times partial derivative with respect to x , and then q -times partial derivative with respect to y is a $(q+p)$ -th order partial derivative of $f(x,y)$, and denoted by $\frac{\partial^{q+p}f}{\partial y^q\partial x^p}(x,y)$.

Formulas.

(i)*Euler's formula.*

$e^{i\theta}=\cos\theta+i\sin\theta$, where θ is any real number.

(ii)*DeMoivre's formula.*

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$, where n is any integer, θ is any real number.

(iii)*Geometric series.*

Suppose z is a complex number, and $|z|<1$. Then $\frac{1}{1-z}=\sum_{k=0}^{\infty}z^k$.

Next, we introduce an important theorem used in this study.

Differentiation term by term theorem ([17]).

If, for all non-negative integer k , the functions $g_k:(a,b)\to R$ satisfy the following three conditions : (i) there exists a point $x_0\in(a,b)$ such that $\sum_{k=0}^{\infty}g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b), (iii) $\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$ is uniformly convergent on (a,b). Then $\sum_{k=0}^{\infty}g_k(x)$ is uniformly convergent and differentiable on (a,b). Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty}g_k(x)=\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$.

Before deriving the major results in this paper, we need a lemma.

Lemma. Suppose z is a complex number, n is any integer, r is a real number, $r \neq 0$ and $|z| \neq |r|$. Then

$$\frac{z^n}{z-r} = -\sum_{k=0}^{\infty}\frac{1}{r^{k+1}}z^{k+n} \qquad \text{if } |z|<|r| \tag{3}$$

$$= \sum_{k=0}^{\infty} r^k z^{-k-1+n} \qquad \text{if } |z|>|r| \tag{4}$$

Proof. If $|z|<|r|$, then $\frac{z^n}{z-r} = -z^n \cdot \frac{1}{r}\cdot\frac{1}{1-\frac{z}{r}}$

$$= -z^n \cdot \frac{1}{r}\cdot\sum_{k=0}^{\infty}\left(\frac{z}{r}\right)^k \qquad \text{(by geometric series)}$$

$$= -\sum_{k=0}^{\infty}\frac{1}{r^{k+1}}z^{k+n}.$$

If $|z|>|r|$, then $\frac{z^n}{z-r} = z^n \cdot \frac{1}{z}\cdot\frac{1}{1-\frac{r}{z}}$

$$= z^n \cdot \frac{1}{z}\cdot\sum_{k=0}^{\infty}\left(\frac{r}{z}\right)^k \qquad \text{(by geometric series)}$$

$$= \sum_{k=0}^{\infty} r^k z^{-k-1+n}$$ ■

The following is the first result in this study, we determine the infinite series forms of any order partial derivatives of two-variables functions (1).

Theorem 1. Suppose r,s,c,d are real numbers, n,p,q are non-negative integers. If the domain of the two-variables function

$$f(x,y) = \frac{y^s\cos[(n-1)(cx+d)] - r\cos[n(cx+d)]}{y^{2s} - 2ry^s\cos(cx+d) + r^2}$$

is $\left\{(x,y)\in R^2 \,\middle|\, y^s exist, |y^s| \neq |r|, y \neq 0\right\}$.

Case(i). If $|y^s|<|r|$, then the $(q+p)$ -th order partial derivative of $f(x,y)$,

$$\frac{\partial^{q+p} f}{\partial y^q \partial x^p}(x,y) = -c^p \sum_{k=0}^{\infty}\frac{(k+n)^p (sk)_q}{r^{k+1}} y^{sk-q}\cos\left[(k+n)(cx+d)+\frac{p\pi}{2}\right] \tag{5}$$

Case(ii). If $|y^s|>|r|$, then

$$\frac{\partial^{q+p} f}{\partial y^q \partial x^p}(x,y) = c^p \sum_{k=0}^{\infty} r^k (-k-1+n)^p (-sk-s)_q y^{-sk-s-q} \cos\left[(-k-1+n)(cx+d)+\frac{p\pi}{2}\right] \quad (6)$$

Proof. In Lemma, we take $z = y^s e^{i(cx+d)}$.

Case(i). If $\left|y^s\right| < |r|$, then by (3) of Lemma, we obtain

$$\frac{[y^s e^{i(cx+d)}]^n}{y^s e^{i(cx+d)} - r} = -\sum_{k=0}^{\infty} \frac{1}{r^{k+1}} [y^s e^{i(cx+d)}]^{k+n}$$

$$\Rightarrow \frac{y^{ns} e^{in(cx+d)}}{[y^s \cos(cx+d) - r] + i y^s \sin(cx+d)} = -\sum_{k=0}^{\infty} \frac{1}{r^{k+1}} y^{(k+n)s} e^{i(k+n)(cx+d)}$$

(by Euler's formula and DeMoivre's formula)

$$\Rightarrow \frac{e^{in(cx+d)}\{[y^s \cos(cx+d) - r] - i y^s \sin(cx+d)\}}{[y^s \cos(cx+d) - r]^2 + [y^s \sin(cx+d)]^2} = -\sum_{k=0}^{\infty} \frac{1}{r^{k+1}} y^{ks} e^{i(k+n)(cx+d)}$$

$$\Rightarrow \frac{\{\cos[n(cx+d)] + i \sin[n(cx+d)]\} \cdot \{[y^s \cos(cx+d) - r] - i y^s \sin(cx+d)\}}{y^{2s} - 2ry^s \cos(cx+d) + r^2} = -\sum_{k=0}^{\infty} \frac{1}{r^{k+1}} y^{sk} e^{i(k+n)(cx+d)} \quad (7)$$

By the equality of real parts of both sides of (7), we have

$$f(x,y) = \frac{y^s \cos[(n-1)(cx+d)] - r\cos[n(cx+d)]}{y^{2s} - 2ry^s \cos(cx+d) + r^2} = -\sum_{k=0}^{\infty} \frac{1}{r^{k+1}} y^{sk} \cos[(k+n)(cx+d)] \quad (8)$$

Therefore, by differentiation term by term, differentiating p-times partial derivative with respect to x, and then q-times partial derivative with respect to y on both sides of (8), we obtain the $(q+p)$-th order partial derivative of $f(x,y)$,

$$\frac{\partial^{q+p} f}{\partial y^q \partial x^p}(x,y) = -c^p \sum_{k=0}^{\infty} \frac{(k+n)^p (sk)_q}{r^{k+1}} y^{sk-q} \cos\left[(k+n)(cx+d)+\frac{p\pi}{2}\right].$$

Case(ii). If $\left|y^s\right| > |r|$, then by (4) of Lemma, we obtain

$$\frac{[y^s e^{i(cx+d)}]^n}{y^s e^{i(cx+d)} - r} = \sum_{k=0}^{\infty} r^k [y^s e^{i(cx+d)}]^{-k-1+n} \quad (9)$$

By the equality of real parts of both sides of (9), we obtain

$$f(x,y) = \frac{y^s \cos[(n-1)(cx+d)] - r\cos[n(cx+d)]}{y^{2s} - 2ry^s \cos(cx+d) + r^2} = \sum_{k=0}^{\infty} r^k y^{(-k-1)s} \cos[(-k-1+n)(cx+d)] \quad (10)$$

Also, by differentiation term by term, differentiating p-times partial derivative with respect to x, and then q-times partial derivative with respect to y on both sides of (10), we have

$$\frac{\partial^{q+p} f}{\partial y^q \partial x^p}(x,y) = c^p \sum_{k=0}^{\infty} r^k (-k-1+n)^p (-sk-s)_q y^{-sk-s-q} \cos\left[(-k-1+n)(cx+d)+\frac{p\pi}{2}\right] \quad \blacksquare$$

Theorem 2. If the assumptions are the same as Theorem 1, and the domain of the two-variables function

$$g(x,y)=\frac{y^s\sin[(n-1)(cx+d)]-r\sin[n(cx+d)]}{y^{2s}-2ry^s\cos(cx+d)+r^2}$$

is $\left\{(x,y)\in R^2 \middle| y^s exist, \left|y^s\right|\neq |r|, y\neq 0\right\}$.

Case(i). If $\left|y^s\right|<|r|$, then the $(q+p)$ -th order partial derivative of $g(x,y)$,

$$\frac{\partial^{q+p}g}{\partial y^q\partial x^p}(x,y)=-c^p\sum_{k=0}^{\infty}\frac{(k+n)^p(sk)_q}{r^{k+1}}y^{sk-q}\sin\left[(k+n)(cx+d)+\frac{p\pi}{2}\right] \tag{11}$$

Case(ii). If $\left|y^s\right|>|r|$, then

$$\frac{\partial^{q+p}g}{\partial y^q\partial x^p}(x,y)=c^p\sum_{k=0}^{\infty}r^k(-k-1+n)^p(-sk-s)_q y^{-sk-s-q}\sin\left[(-k-1+n)(cx+d)+\frac{p\pi}{2}\right] \tag{12}$$

Proof. Case(i). If $\left|y^s\right|<|r|$, then by the equality of imaginary parts of both sides of (7), we obtain

$$g(x,y)=\frac{y^s\sin[(n-1)(cx+d)]-r\sin[n(cx+d)]}{y^{2s}-2ry^s\cos(cx+d)+r^2}=-\sum_{k=0}^{\infty}\frac{1}{r^{k+1}}y^{sk}\sin[(k+n)(cx+d)] \tag{13}$$

Therefore, by differentiation term by term, differentiating p -times partial derivative with respect to x, and then q -times partial derivative with respect to y on both sides of (13), we obtain the $(q+p)$ -th order partial derivative of $g(x,y)$,

$$\frac{\partial^{q+p}g}{\partial y^q\partial x^p}(x,y)=-c^p\sum_{k=0}^{\infty}\frac{(k+n)^p(sk)_q}{r^{k+1}}y^{sk-q}\sin\left[(k+n)(cx+d)+\frac{p\pi}{2}\right].$$

Case(ii). If $\left|y^s\right|>|r|$, then by the equality of imaginary parts of both sides of (9),

$$g(x,y)=\frac{y^s\sin[(n-1)(cx+d)]-r\sin[n(cx+d)]}{y^{2s}-2ry^s\cos(cx+d)+r^2}=\sum_{k=0}^{\infty}r^k y^{(-k-1)s}\sin[(-k-1+n)(cx+d)] \tag{14}$$

Also, by differentiation term by term, differentiating p -times partial derivative with respect to x, and then q -times partial derivative with respect to y on both sides of (14), we obtain the $(q+p)$ -th order partial derivative of $g(x,y)$,

$$\frac{\partial^{q+p}g}{\partial y^q\partial x^p}(x,y)=c^p\sum_{k=0}^{\infty}r^k(-k-1+n)^p(-sk-s)_q y^{-sk-s-q}\sin\left[(-k-1+n)(cx+d)+\frac{p\pi}{2}\right]$$ ■

III. EXAMPLES

In the following, we propose two examples of two-variables functions to determine their any order partial derivatives, and some of their higher order partial derivative values practically. On the other hand, we use Maple to calculate the approximations of these higher order partial derivative values and their infinite series forms for verifying our answers.

Example 1. If the domain of the two-variables function

$$f(x,y) = \frac{y^3 \cos[4(2x-3)] - 7\cos[5(2x-3)]}{y^6 - 14y^3\cos(2x-3) + 49} \quad (15)$$

is $\{(x,y) \in R^2 \,\|y^3| \neq 7, y \neq 0\}$.

Case(i). If $|y^3| < 7$ and $y \neq 0$, then by (5) of Theorem 1, we obtain any $(q+p)$ -th order partial derivative of $f(x,y)$,

$$\frac{\partial^{q+p} f}{\partial y^q \partial x^p}(x,y) = -2^p \sum_{k=0}^{\infty} \frac{(k+5)^p (3k)_q}{7^{k+1}} y^{3k-q} \cos\left[(k+5)(2x-3) + \frac{p\pi}{2}\right] \quad (16)$$

Thus, we can evaluate the 7-th order partial derivative value of $f(x,y)$ at $\left(\frac{1}{2},\frac{1}{3}\right)$,

$$\frac{\partial^7 f}{\partial y^4 \partial x^3}\left(\frac{1}{2},\frac{1}{3}\right) = -8\sum_{k=0}^{\infty} \frac{(k+5)^3 (3k)_4}{7^{k+1}} \left(\frac{1}{3}\right)^{3k-4} \sin(-2k-10) \quad (17)$$

Next, we use Maple to verify the correctness of (17).

```
>f:=(x,y)->(y^3*cos(4*(2*x-3))-7*cos(5*(2*x-3)))/(y^6-14*y^3*cos(2*x-3)+49);
```

$$f := (x,y) \to \frac{y^3\cos(8x-12) - 7\cos(10x-15)}{y^6 - 14y^3\cos(2x-3) + 49}$$

```
>evalf(D[1$3,2$4](f)(1/2,1/3),14);
```

310.42188432390

```
>evalf(-8*sum((k+5)^3*product(3*k-j,j=0..3)*(1/3)^(3*k-4)/7^(k+1)*sin(-2*k-10),k=0..infinity),14);
```

310.42188432389

Case(ii). If $|y^3| > 7$, then by (6) of Theorem 1, we obtain

$$\frac{\partial^{q+p} f}{\partial y^q \partial x^p}(x,y) = 2^p \sum_{k=0}^{\infty} 7^k (-k+4)^p (-3k-3)_q y^{-3k-3-q} \cos\left[(-k+4)(2x-3) + \frac{p\pi}{2}\right] \quad (18)$$

Hence, we can evaluate the 9-th order partial derivative value of $f(x,y)$ at $(1,2)$,

$$\frac{\partial^9 f}{\partial y^5 \partial x^4}(1,2) = 16\sum_{k=0}^{\infty} 7^k (-k+4)^4 (-3k-3)_5 2^{-3k-8} \cos(k-4) \quad (19)$$

We also use Maple to verify the correctness of (19).

```
>evalf(D[1$4,2$5](f)(1,2),20);
```

$$-2.7262267806502608761 \cdot 10^6$$

```
>evalf(16*sum(7^k*(-k+4)^4*product(-3*k-3-j,j=0..4)*2^(-3*k-8)*cos(k-4),k=0..infinity),20);
```

$$-2.7262267806502586403 \cdot 10^6$$

Example 2. If the domain of the two-variables function

$$g(x,y) = \frac{y^5 \sin[5(4x+2)] + 4\sin[6(4x+2)]}{y^{10} + 8y^5\cos(4x+2) + 16} \tag{20}$$

is $\left\{(x,y) \in R^2 \middle| \left|y^5\right| \neq 4,\ y \neq 0\right\}$.

Case(i). If $\left|y^5\right| < 4$ and $y \neq 0$, then by (11) of Theorem 2, we obtain any $(q+p)$ -th order partial derivative of $g(x,y)$,

$$\frac{\partial^{q+p} g}{\partial y^q \partial x^p}(x,y) = -4^p \sum_{k=0}^{\infty} \frac{(k+6)^p (5k)_q}{(-4)^{k+1}} y^{5k-q} \sin\left[(k+6)(4x+2) + \frac{p\pi}{2}\right] \tag{21}$$

Therefore, we can determine the 8-th order partial derivative value of $g(x,y)$ at $\left(-\frac{1}{4}, \frac{1}{5}\right)$,

$$\frac{\partial^8 g}{\partial y^4 \partial x^4}\left(-\frac{1}{4}, \frac{1}{5}\right) = -256 \sum_{k=0}^{\infty} \frac{(k+6)^4 (5k)_4}{(-4)^{k+1}} \left(\frac{1}{5}\right)^{5k-4} \sin(k+6) \tag{22}$$

In the following, we use Maple to verify the correctness of (22).

>g:=(x,y)->(y^5*sin(5*(4*x+2))+4*sin(6*(4*x+2)))/(y^10+8*y^5*cos(4*x+2)+16);

$$g := (x,y) \rightarrow \frac{y^5 \sin(20x+10) + 4\sin(24x+12)}{y^{10} + 8y^5\cos(4x+2) + 16}$$

>evalf(D[1$4,2$4](g)(-1/4,1/5),14);

$$-6.0050436413231 \cdot 10^5$$

>evalf(-256*sum((k+6)^4*product(5*k-j,j=0..3)*(1/5)^(5*k-4)/(-4)^(k+1)*sin(k+6),k=0..infinity),14);

$$-6.0050436413230 \cdot 10^5$$

Case(ii). If $\left|y^5\right| > 4$, then by (12) of Theorem 2, we obtain

$$\frac{\partial^{q+p} g}{\partial y^q \partial x^p}(x,y) = 4^p \sum_{k=0}^{\infty} (-4)^k (-k+5)^p (-5k-5)_q y^{-5k-5-q} \sin\left[(-k+5)(4x+2) + \frac{p\pi}{2}\right] \tag{23}$$

Hence, we can evaluate the 9-th order partial derivative value of $g(x,y)$ at $\left(-\frac{3}{4}, -3\right)$,

$$\frac{\partial^9 g}{\partial y^6 \partial x^3}\left(-\frac{3}{4}, -3\right) = -64 \sum_{k=0}^{\infty} (-4)^k (-k+5)^3 (-5k-5)_6 (-3)^{-5k-11} \cos(k-5) \tag{24}$$

Using Maple to verify the correctness of (24) as follows:

>evalf(D[1$3,2$6](g)(-3/4,-3),28);

966.6931679522767311583189 8

>evalf(-64*sum((-4)^k*(-k+5)^3*product(-5*k-5-j,j=0..5)*(-3)^(-5*k-11)*cos(k-5),k=0..infinity),28);

966.6931679522767311583314549

IV. CONCLUSIONS

From the above discussion, we know the differentiation term by term plays a significant role in the theoretical inferences of this study. In fact, the application of this theorem is extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications.

On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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