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RESEARCH ARTICLE

A Second-Order Linear Ordinary Differential Equation (ODE)

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Abstract

The term "hypergeometric function" sometimes refers to the generalized hypergeometric function.

In mathematics, the Gaussian or ordinary hypergeometric function $_2F_1(a,b;c;z)$ is a special function represented by the hypergeometric series, that includes many other special functions as special or limiting cases. It is a solution of a second-order linear ordinary differential equation (ODE). Every second-order linear ODE with three regular singular points can be transformed into this equation

Keywords….. Hypergeometric function

Introduction

The term "hypergeometric series" was first used by John Wallis in his 1655 book Arithmetica Infinitorum. Hypergeometric series were studied by Euler, but the first full systematic treatment was given by Gauss (1813), Studies in the nineteenth century included those of Ernst Kummer (1836), and the fundamental characterisation by Bernhard Riemann of the hypergeometric function by means of the differential equation it satisfies. Riemann showed that the second-order differential equation (in z) for the $_2F_1$, examined in the complex plane, could be characterised (on the Riemann sphere) by its three regular singularities.

The cases where the solutions are algebraic functions were found by H. A. Schwarz (Schwarz's list).

In this paper, we will discuss these main things

1- **Hypergeometric function**
2- **Generalized hypergeometric function**
3- **Basic hypergeometric series**
4- **Confluent hypergeometric function**

Hypergeometric functions are an important tool in many branches of pure and applied mathematics, and they encompass most special functions, including the Chebyshev polynomials. There are also well-known connections between Chebyshev polynomials and sequences of numbers and polynomials related to Fibonacci numbers. However, to my knowledge and with one small exception, direct connections between Fibonacci numbers and hypergeometric functions have not been established or exploited before.
It is the purpose of this paper to give a brief exposition of hypergeometric functions, as far as is relevant to the Fibonacci and allied sequences. A variety of representations in terms of infinite sums and infinite series involving binomial coefficients are obtained. While many of them are well-known, some identities appear to be new.

The method of hypergeometric functions works just as well for other sequences, especially the Lucas, Pell, and associated Pell numbers and polynomials, and also for more general second-order linear recursion sequences. However, apart from the _nal section, we will restrict our attention to Fibonacci numbers as the most prominent example of a second order recurrence.(KARL DILCHER AND LEONARDO PISANO)

1-Hypergeometric Functions

Almost all of the most common special functions in mathematics and mathematical physics are particular cases of the Gauss hypergeometric series defined by

$$2F1\,(a; b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{z^k}{k!}$$

where the rising factorial (a)k is defined by (a)0 = 1 and

(a)k = a(a + 1) ….. (a + k —1); (k >= 1),

For arbitrary a € C . The series (2.1) is not defined when c = - m, with m = 0,1,2……, unless a or b are equal to -n, n = 0; 1; 2; : : :, and n < m. It is also easy to see that the series (2.1) reduces to a polynomial of degree n in z when a or b is equal to □n, n = 0,1,2,…... In all other cases the series has radius of convergence 1; this follows from the ratio test and (2.2). The function defined by the series (2.1) is called the Gauss hypergeometric function. When there is no danger of confusion with other types of hypergeometric series, (2.1) is commonly denoted simply by F(a; b; c; z) and called the hypergeometric series, resp. function.

Most properties of the hypergeometric series can be found in the well-known reference works [1], [9] and [8] (in increasing order of completeness). Proofs of many of the more important properties can be found, e.g., in [10]; see also the important works [4] and [11].
At this point we mention only the special case
$F(a; b; b; z) = (1 - z)^{-a}$, the binomial formula. The case a = 1 yields the geometric series; this gave rise to the term hypergeometric.
More properties will be introduced in later sections, as the need arises.

2-generalized hypergeometric function

@@ A generalized hypergeometric function $_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}\,x.$$

(The factor of $k+1$ in the denominator is present for historical reasons of notation.)
The function $_2F_1(a, b; c; x)$ corresponding to $p=2$, $q=1$ is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions to the hypergeometric differential equation, which has a regular singular point at the origin. To derive the hypergeometric function from the hypergeometric differential equation .

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0,$$

use the Frobenius method to reduce it to

$$\sum_{n=0}^{\infty}\left\{(n+1)(n+c)A_{n+1} - \left[n^2+(a+b)n+ab\right]A_n\right\}z^n = 0,$$

giving the indicial equation

$$A_{n+1} = \frac{(n+a)(n+b)}{(n+1)(n+c)}\,A_n.$$

Plugging this into the ansatz series

$$y = \sum_{n=0}^{\infty} A_n z^n$$

then gives the solution

$$y = A_0\left[1 + \frac{ab}{1!\,c}\,z + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}\,z^2 + \ldots\right].$$

This is the so-called regular solution, denoted

$$
\begin{aligned}
{}_2F_1(a, b; c; z) &= 1 + \frac{a\,b}{1!\,c}\,z + \frac{a\,(a+1)\,b\,(b+1)}{2!\,c\,(c+1)}\,z^2 + \ldots \\
&= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!},
\end{aligned}
$$

which converges if c is not a negative integer (1) for all of $|z| < 1$ and (2) on the unit circle $|z| = 1$ if $\Re[c - a - b] > 0$. Here, $(a)_n$ is a Pochhammer symbol.

The complete solution to the hypergeometric differential equation is

$$y = A\,{}_2F_1(a, b; c; z) + B\,z^{1-c}\,{}_2F_1(a+1-c, b+1-c; 2-c; z).$$

The hypergeometric series is convergent for *arbitrary* a, b, and c for real $-1 < z < 1$, and for $z = \pm 1$ if $c > a + b$.

Derivatives of ${}_2F_1(a, b; c; z)$ are given by

$$
\begin{aligned}
\frac{d\,{}_2F_1(a, b; c; z)}{d\,z} &= \frac{a\,b}{c}\,{}_2F_1(a+1, b+1; c+1; z) \\
\frac{d^2\,{}_2F_1(a, b; c; z)}{d\,z^2} &= \frac{a\,(a+1)\,b\,(b+1)}{c\,(c+1)}\,{}_2F_1(a+2, b+2; c+2; z)
\end{aligned}
$$

(Magnus and Oberhettinger 1949, p. 8).

Hypergeometric functions with special arguments reduce to elementary functions, for example,

$$
\begin{aligned}
{}_2F_1(1, 1; 1; z) &= \frac{1}{1-z} \\
{}_2F_1(1, 1; 2; z) &= -\frac{\ln(1-z)}{z} \\
{}_2F_1(1, 2; 1; z) &= \frac{1}{(1-z)^2} \\
{}_2F_1(1, 2; 2; z) &= \frac{1}{1-z}.
\end{aligned}
$$

An integral giving the hypergeometric function is

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 \frac{t^{b-1}\,(1-t)^{c-b-1}}{(1-t\,z)^a}\,d\,t$$

As shown by Euler in 1748 (Bailey 1935, pp. 4-5). Barnes (1908) gave the contour integral:

$${}_2F_1(a, b; c; z) = \frac{1}{2\,\pi\,i} \int_{-i\,\infty}^{i\,\infty} \frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(-s)}{\Gamma(c-s)}\,(-z)^s\,d\,s,$$

where $|\arg(-z)| < \pi$ and the path is curved (if necessary) to separate the poles $s = -a - n$, $s = -b - n$, ... ($n = 0$, 1, ...) from the poles $s = 0$, 1 ... (Bailey 1935, pp. 4-5; Whittaker and Watson 1990).

Curiously, at a number of very special points, the hypergeometric functions can assume rational,

$$
\begin{aligned}
{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{5}{6}; \tfrac{27}{32}\right) &= \tfrac{8}{5} \\
{}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{2}; \tfrac{3}{4}; \tfrac{80}{81}\right) &= \tfrac{9}{5}
\end{aligned}
$$

(M. Trott, pers. comm., Aug. 5, 2002; Zucker and Joyce 2001), quadratic surd

$$ {}_2F_1\left(\tfrac{1}{8},\tfrac{3}{8};\tfrac{1}{2};\tfrac{2400}{2401}\right) = \tfrac{2}{3}\sqrt{7} $$
$$ {}_2F_1\left(\tfrac{1}{6},\tfrac{1}{3};\tfrac{1}{2};\tfrac{25}{27}\right) = \tfrac{3}{4}\sqrt{3} $$

(Zucker and Joyce 2001), and other exact values

$$ {}_2F_1\left(\tfrac{1}{6},\tfrac{1}{2};\tfrac{2}{3};\tfrac{125}{128}\right) = \tfrac{4}{3}\,2^{1/6} $$
$$ {}_2F_1\left(\tfrac{1}{12},\tfrac{5}{12};\tfrac{1}{2};\tfrac{1323}{1331}\right) = \tfrac{3}{4}\,(11)^{1/4} $$
$$ {}_2F_1\left(\tfrac{1}{12},\tfrac{5}{12};\tfrac{1}{2};\tfrac{121}{125}\right) = \frac{2^{1/6}\,(15)^{1/4}}{4\sqrt{\pi}}\,\frac{\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}\left(1+\sqrt{3}\right) $$

(Zucker and Joyce 2001, 2003).

An infinite family of rational values for well-poised hypergeometric functions with rational arguments is given by;

$$ {}_kF_{k-1}\left(\frac{1}{k+1},\ldots,\frac{k}{k+1};\frac{2}{k},\frac{3}{k},\ldots,\frac{k-1}{k},\frac{k+1}{k};\left(\frac{x\left(1-x^k\right)}{f_k}\right)^k\right) = \frac{1}{1-x^k} $$

for $k=2$**, 3, ...,** $0\le x\le (k+1)^{-1/k}$**, and**

$$ f_k \equiv \frac{k}{(1+k)^{1+1/k}} $$

(M. L. Glasser, pers. comm., Sept. 26, 2003). This gives the particular identity

$$ {}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};\tfrac{3}{2};\tfrac{27}{4}x^2\left(1-x^2\right)^2\right) $$
$$ = \frac{2\sin\left[\frac{1}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3}\,x\left(1-x^2\right)\right)\right]}{\sqrt{3}\,x\left(1-x^2\right)} $$
$$ = \frac{1}{1-x^2} $$

for $0\le x\le \sqrt{3}/3$**.**

A hypergeometric function can be written using <u>Euler's hypergeometric transformations</u>

$$ \begin{array}{lll} t & \to & t \\ t & \to & 1-t \\ t & \to & (1-z-tz)^{-1} \\ t & \to & \dfrac{1-t}{1-tz} \end{array} $$

in any one of four equivalent forms

$$ \begin{aligned} {}_2F_1(a,b;c;z) &= (1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right) \\ &= (1-z)^{-b}\,{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right) \\ &= (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z) \end{aligned} $$

(Abramowitz and Stegun 1972, p. 559).

It can also be written as a linear combination

$$
{}_2F_1(a, b; c; z) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,{}_2F_1(a, b; a+b+1-c; 1-z)
+ \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,(1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; 1+c-a-b; 1-z)
$$

(Barnes 1908; Bailey 1935, pp. 3-4; Whittaker and Watson 1990, p. 291).

Kummer found all six solutions (not necessarily regular at the origin) to the hypergeometric differential equation:

$$
\begin{aligned}
u_1(x) &= {}_2F_1(a, b; c; z) \\
u_2(x) &= {}_2F_1(a, b; a+b+1-c; 1-z) \\
u_3(x) &= z^{-a}\,{}_2F_1\left(a, a+1-c; a+1-b; z^{-1}\right) \\
u_4(x) &= z^{-b}\,{}_2F_1\left(b+1-c, b; b+1-a; z^{-1}\right) \\
u_5(x) &= z^{1-c}\,{}_2F_1(b+1-c, a+1-c; 2-c; z) \\
u_6(x) &= (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c+1-a-b; 1-z)
\end{aligned}
$$

(Abramowitz and Stegun 1972, p. 563).

Applying Euler's hypergeometric transformations to the Kummer solutions then gives all 24 possible forms which are solutions to the hypergeometric differential equation:

$$
\begin{aligned}
u_1^{(1)}(x) &= {}_2F_1(a, b; c; z) \\
u_1^{(2)}(x) &= (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z) \\
u_1^{(3)}(x) &= (1-z)^{-a}\,{}_2F_1(a, c-b; c; z/(z-1)) \\
u_1^{(4)}(x) &= (1-z)^{-b}\,{}_2F_1(c-a, b; c; z/(z-1)) \\
u_2^{(1)}(x) &= {}_2F_1(a, b; a+b+1-c; 1-z) \\
u_2^{(2)}(x) &= z^{1-c}\,{}_2F_1(a+1-c, b+1-c; a+b+1-c; 1-z) \\
u_2^{(3)}(x) &= z^{-a}\,{}_2F_1\left(a, a+1-c; a+b+1-c; 1-z^{-1}\right) \\
u_2^{(4)}(x) &= z^{-b}\,{}_2F_1\left(b+1-c, b; a+b+1-c; 1-z^{-1}\right) \\
u_3^{(1)}(x) &= (-z)^{-a}\,{}_2F_1\left(a, a+1-c; a+1-b; z^{-1}\right) \\
u_3^{(2)}(x) &= (-z)^{b-c}(1-z)^{c-a-b}\,{}_2F_1\left(1-b, c-b; a+1-b; z^{-1}\right) \\
u_3^{(3)}(x) &= (1-z)^{-a}\,{}_2F_1\left(a, c-b; a+1-b; (1-z)^{-1}\right) \\
u_3^{(4)}(x) &= (-z)^{1-c}(1-z)^{c-a-1}\,{}_2F_1\left(a+1-c, 1-b; a+1-b; (1-z)^{-1}\right) \\
u_4^{(1)}(x) &= (-z)^{-b}\,{}_2F_1\left(b+1-c, b; b+1-a; z^{-1}\right) \\
u_4^{(2)}(x) &= (-z)^{a-c}(1-z)^{c-a-b}\,{}_2F_1\left(1-a, c-a; b+1-a; z^{-1}\right) \\
u_4^{(3)}(x) &= (1-z)^{-b}\,{}_2F_1\left(b, c-a; b+1-a; (1-z)^{-1}\right) \\
u_4^{(4)}(x) &= (-z)^{1-c}(1-z)^{c-b-1}\,{}_2F_1\left(b+1-c, 1-a; b+1-a; (1-z)^{-1}\right) \\
u_5^{(1)}(x) &= z^{1-c}\,{}_2F_1(a+1-c, b+1-c; 2-c; z) \\
u_5^{(2)}(x) &= z^{1-c}(1-z)^{c-a-b}\,{}_2F_1(1-a, 1-b; 2-c; z) \\
u_5^{(3)}(x) &= z^{1-c}(1-z)^{c-a-1}\,{}_2F_1(a+1-c, 1-b; 2-c; z/(z-1)) \\
u_5^{(4)}(x) &= z^{1-c}(1-z)^{c-b-1}\,{}_2F_1(b+1-c, 1-a; 2-c; z/(z-1)) \\
u_6^{(1)}(x) &= (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c+1-a-b; 1-z) \\
u_6^{(2)}(x) &= z^{1-c}(1-z)^{c-a-b}\,{}_2F_1(1-a, 1-b; c+1-a-b; 1-z) \\
u_6^{(3)}(x) &= z^{a-c}(1-z)^{c-a-b}\,{}_2F_1\left(c-a, 1-a; c+1-a-b; 1-z^{-1}\right)
\end{aligned}
$$

$$u_6^{(4)}(x) = z^{b-c}(1-z)^{c-a-b}\,{}_2F_1\left(c-b, 1-b; c+1-a-b; 1-z^{-1}\right)$$

(Kummer 1836; Erdélyi *et al.* 1981, pp. 105-106).

Goursat (1881) and Erdélyi *et al.* (1981) give many hypergeometric transformation formulas, including several cubic transformations.

Many functions of mathematical physics can be expressed as special cases of the hypergeometric functions. For example,

$${}_2F_1(-l, l+1; 1; (1-z)/2) = P_l(z),$$

where $P_l(z)$ **is a Legendre polynomial.**

$$(1+z)^n = {}_2F_1(-n, b; b; -z)$$

$$\ln(1+z) = z\,{}_2F_1(1, 1; 2; -z)$$

Complete elliptic integrals and the Riemann P-series can also be expressed in terms of ${}_2F_1(a, b; c; z)$**. Special values include**

$${}_2F_1(a, b; a-b+1; -1) = 2^{-a}\sqrt{\pi}\,\frac{\Gamma(1+a+b)}{\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)}$$

$${}_2F_1(1, -a; a; -1) = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(a)}{\Gamma\left(a+\frac{1}{2}\right)} + 1$$

$${}_2F_1\left(a, b; c; \tfrac{1}{2}\right) = 2^a\,{}_2F_1(a, c-b; c; -1)$$

$${}_2F_1\left(a, b; \tfrac{1}{2}(a+b+1); \tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left[\frac{1}{2}(1+a+b)\right]}{\Gamma\left[\frac{1}{2}(1+a)\right]\Gamma\left[\frac{1}{2}(1+b)\right]}$$

$${}_2F_1\left(a, 1-a; c; \tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left[\frac{1}{2}(c+1)\right]}{\Gamma\left[\frac{1}{2}(a+c)\right]\Gamma\left[\frac{1}{2}(1+c-a)\right]}$$

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

Kummer's first formula gives

$${}_2F_1\left(\tfrac{1}{2}+m-k, -n; 2m+1; 1\right) = \frac{\Gamma(2m+1)\,\Gamma\left(m+\frac{1}{2}+k+n\right)}{\Gamma\left(m+\frac{1}{2}+k\right)\Gamma(2m+1+n)},$$

where $m \neq -1/2$**,** -1**,** $-3/2$**, ... Many additional identities are given by Abramowitz and Stegun (1972, p. 557).**

Hypergeometric functions can be generalized to generalized hypergeometric functions

$${}_nF_m(a_1, \ldots, a_n; b_1, \ldots, b_m; z).$$

A function of the form ${}_1F_1(a; b; z)$ **is called a confluent hypergeometric function of the first kind, and a function of the form** ${}_0F_1(; b; z)$ **is called a confluent hypergeometric limit function.**

3- Basic hypergeometric series

In mathematics, Heine's basic hypergeometric series, or hypergeometric q-series, are q-analog generalizations of generalized hypergeometric series, and are in turn generalized by elliptic hypergeometric series. A series x_n is called hypergeometric if the ratio of successive terms x_{n+1}/x_n is a rational function of n. If the ratio of successive terms is a rational function of q^n, then the series is called a basic hypergeometric series. The number q is called the base.

The basic hypergeometric series ${}_2\varphi_1(q^\alpha,q^\beta;q^\gamma;q,x)$ was first considered by Eduard Heine (1846). It becomes the hypergeometric series $F(\alpha,\beta;\gamma;x)$ in the limit when the base q is 1

&Definition

There are two forms of basic hypergeometric series, the unilateral basic hypergeometric series φ, and the more general bilateral basic geometric series ψ. The unilateral basic hypergeometric series is defined as

$$ {}_j\phi_k\left[\begin{matrix} a_1 & a_2 & \ldots & a_j \\ b_1 & b_2 & \ldots & b_k \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_j; q)_n}{(b_1, b_2, \ldots, b_k, q; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+k-j} z^n $$

where

$$ (a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \ldots (a_m; q)_n $$

and where

$$ (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}). $$

is the q-shifted factorial. The most important special case is when $j = k+1$, when it becomes

$$ {}_{k+1}\phi_k\left[\begin{matrix} a_1 & a_2 & \ldots & a_k & a_{k+1} \\ b_1 & b_2 & \ldots & b_k & \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{k+1}; q)_n}{(b_1, b_2, \ldots, b_k, q; q)_n} z^n. $$

This series is called balanced if $a_1 \ldots a_{k+1} = b_1 \ldots b_k q$. This series is called well poised if $a_1 q = a_2 b_1 = \ldots = a_{k+1} b_k$, and very well poised if in addition $a_2 = -a_3 = q a_1^{1/2}$.

The bilateral basic hypergeometric series, corresponding to the bilateral hypergeometric series, is defined as

$$ {}_j\psi_k\left[\begin{matrix} a_1 & a_2 & \ldots & a_j \\ b_1 & b_2 & \ldots & b_k \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_j; q)_n}{(b_1, b_2, \ldots, b_k; q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{k-j} z^n. $$

The most important special case is when $j = k$, when it becomes

$$ {}_k\psi_k\left[\begin{matrix} a_1 & a_2 & \ldots & a_k \\ b_1 & b_2 & \ldots & b_k \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_k; q)_n}{(b_1, b_2, \ldots, b_k; q)_n} z^n. $$

The unilateral series can be obtained as a special case of the bilateral one by setting one of the b variables equal to q, at least when none of the a variables is a power of q., as all the terms with n<0 then vanish.

& Simple series

Some simple series expressions include

$$\frac{z}{1-q}\ {}_2\phi_1\left[\begin{matrix}q\ q\\ q^2\end{matrix};q,z\right]=\frac{z}{1-q}+\frac{z^2}{1-q^2}+\frac{z^3}{1-q^3}+\ldots$$

and

$$\frac{z}{1-q^{1/2}}\ {}_2\phi_1\left[\begin{matrix}q\ q^{1/2}\\ q^{3/2}\end{matrix};q,z\right]=\frac{z}{1-q^{1/2}}+\frac{z^2}{1-q^{3/2}}+\frac{z^3}{1-q^{5/2}}+\ldots$$

and

$${}_2\phi_1\left[\begin{matrix}q\ \ -1\\ -q\end{matrix};q,z\right]=1+\frac{2z}{1+q}+\frac{2z^2}{1+q^2}+\frac{2z^3}{1+q^3}+\ldots.$$

& The q-binomial theorem

The *q*-binomial theorem states that

$${}_1\phi_0(a;q,z)=\frac{(az;q)_\infty}{(z;q)_\infty}=\prod_{n=0}^{\infty}\frac{1-aq^nz}{1-q^nz}$$

which follows by repeatedly applying the identity

$${}_1\phi_0(a;q,z)=\frac{1-az}{1-z}\ {}_1\phi_0(a;q,qz).$$

The special case of $a = 0$ is closely related to the q-exponential.

& Ramanujan's identity

Ramanujan gave the identity

valid for $|q| < 1$ and $|b/a| < |z| < 1$. Similar identities for ${}_6\psi_6$ have been given by Bailey. Such identities can be understood to be generalizations of the Jacobi triple product theorem, which can be written using q-series as

$$\sum_{n=-\infty}^{\infty}q^{n(n+1)/2}z^n=(q;q)_\infty\ (-1/z;q)_\infty\ (-zq;q)_\infty.$$

Ken Ono gives a related formal power series

$$A(z;q)\stackrel{\text{def}}{=}\frac{1}{1+z}\sum_{n=0}^{\infty}\frac{(z;q)_n}{(-zq;q)_n}z^n=\sum_{n=0}^{\infty}(-1)^nz^{2n}q^{n^2}.$$

&Watson's contour integral

As an analogue of the Barnes integral for the hypergeometric series, Watson showed that

$${}_2\phi_1(a,b;c;q,z)=\frac{-1}{2\pi i}\frac{(a,b;q)_\infty}{(q,c;q)_\infty}\int_{-i\infty}^{i\infty}\frac{(qq^s,cq^s;q)_\infty}{(aq^s,bq^s;q)_\infty}\frac{\pi(-z)^s}{\sin\pi s}ds$$

where the poles of $(aq^s,bq^s;q)_\infty$ lie to the left of the contour and the remaining poles lie to the right. There is a similar contour integral for ${}_{r+1}\varphi_r$. This contour integral gives an analytic continuation of the basic hypergeometric function in *z*.

4-Confluent hypergeometric function

In mathematics, a confluent hypergeometric function is a solution of a confluent hypergeometric equation, which is a degenerate form of a hypergeometric differential equation where two of the three regular singularities merge into an irregular singularity. (The term "confluent" refers to the merging of singular points of families of differential equations; "confluere" is Latin for "to flow together".) There are several common standard forms of confluent hypergeometric functions:

- **Kummer's (confluent hypergeometric) function $M(a,b,z)$, introduced by Kummer (1837), is a solution to Kummer's differential equation. There is a different but unrelated Kummer's function bearing the same name.**
- **Whittaker functions (for Edmund Taylor Whittaker) are solutions to Whittaker's equation.**
- **Coulomb wave functions are solutions to Coulomb wave equation.**

The Kummer functions, Whittaker functions, and Coulomb wave functions are essentially the same, and differ from each other only by elementary functions and change of variables.

&Kummer's equation

Kummer's equation is

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw=0.,$$

with a regular singular point at 0 and an irregular singular point at ∞. It has two linearly independent solutions $M(a,b,z)$ and $U(a,b,z)$.

Kummer's function (of the first kind) M is a generalized hypergeometric series introduced in (Kummer 1837), given by

$$M(a,b,z)=\sum_{n=0}^{\infty}\frac{(a)_n z^n}{(b)_n n!}={}_1F_1(a;b;z)$$

where

$$(a)_n=a(a+1)(a+2)\cdots(a+n-1)$$

is the rising factorial. Another common notation for this solution is $\Phi(a,b,z)$. This defines an entire function of $a.b$, and z, except for poles at $b = 0, -1, -2, ...$

Just as the confluent differential equation is a limit of the hypergeometric differential equation as the singular point at 1 is moved towards the singular point at ∞, the confluent hypergeometric function can be given as a limit of the hypergeometric function

$$M(a,c,z)=\lim_{b\to\infty}{}_2F_1(a,b;c;z/b)$$

and many of the properties of the confluent hypergeometric function are limiting cases of properties of the hypergeometric function.

Another solution of Kummer's equation is the Tricomi confluent hypergeometric function $U(a;b;z)$ introduced by Francesco Tricomi (1947), and sometimes denoted by $\Psi(a;b;.z)$. The function U is defined in terms of Kummer's function M by

$$U(a,b,z)=\frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b,z)+\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(a-b+1,2-b,z).$$

This is undefined for integer *b*, but can be extended to integer *b* by continuity.

& Integral representations

For certain values of *a* and *b*, *M*(*a*,*b*,*z*) can be represented as an integral

$$M(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}\,du \quad \mathrm{re}\ b>\mathrm{re}\ a>0.$$

For *a* with positive real part *U* can be obtained by the Laplace integral

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}\,dt, \quad (\mathrm{re}\ a>0)$$

The integral defines a solution in the right half-plane re *z* > 0.

They can also be represented as Barnes integrals

$$M(a,b,z)=\frac{1}{2\pi i}\frac{\Gamma(b)}{\Gamma(a)}\int_{-i\infty}^{i\infty}\frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(b+s)}(-z)^s ds$$

where the contour passes to one side of the poles of Γ(−*s*) and to the other side of the poles of Γ(*a*+*s*).

& Asymptotic behavior

The asymptotic behavior of *U*(*a*,*b*,*z*) as $z\to\infty$ can be deduced from the integral representations. If *z* = *x* is real, then making a change of variables in the integral followed by expanding the binomial series and integrating it formally term by term gives rise to an asymptotic series expansion, valid as $x\to\infty$:[1]

$$U(a,b,x)\sim x^{-a}\,{}_2F_0\left(a,a-b+1;\,;-\frac{1}{x}\right),$$

where ${}_2F_0(\cdot,\cdot;\,;-1/x)$ is a generalized hypergeometric series, which converges nowhere but exists as a formal power series in 1/*x*.

& Relations

There are many relations between Kummer functions for various arguments and their derivatives. This section gives a few typical examples.

& Contiguous relations

Given $M(a,b;z)$, the four functions $M(a\pm1,b;z)$, $M(a,b\pm1;z)$ are called contiguous to $M(a,b;z)$. The function $M(a,b;z)$ can be written as a linear combination of any two of its contiguous functions, with rational coefficients in terms of *a*, *b* and *z*. This gives (4,2)=6 relations, given by identifying any two lines on the right hand side of

$$\begin{aligned} z\frac{dM}{dz}=z\frac{a}{b}M(a+,b+) &= a(M(a+)-M)\\ &=(b-1)(M(b-)-M)\\ &=(b-a)M(a-)+(a-b+z)M\\ &=z(a-b)M(b+)/b+zM \end{aligned}$$

In the notation above, $M=M(a,b;z),M(a+)=M(a+1,b;z)$ and so on.

Repeatedly applying these relations gives a linear relation between any three functions of the form $M(a+m,b+n;z)$(and their higher derivatives), where *m*, *n* are integers. There are similar relations for *U*.

Kummer's transformation

Kummer's functions are also related by Kummer's transformations:

$$M(a,b,z) = e^z\,M(b-a,b,-z)$$

$$U(a,b,z) = z^{1-b}U\left(1+a-b,2-b,z\right).$$

Multiplication theorem

The following multiplication theorems hold true:

$$U(a,b,z) = e^{(1-t)z}\sum_{i=0}\frac{(t-1)^i z^i}{i!}U(a,b+i,zt) =$$

$$= e^{(1-t)z}t^{b-1}\sum_{i=0}\frac{\left(1-\frac{1}{t}\right)^i}{i!}U(a-i,b-i,zt).$$

Connection with Laguerre polynomials and similar representations

In terms of Laguerre polynomials, Kummer's functions have several expansions, for example

$$M\left(a,b,\frac{xy}{x-1}\right) = (1-x)^a\cdot\sum_n\frac{(a)_n}{(b)_n}L_n^{(b-1)}(y)x^n$$ **(Erdelyi 1953, 6.12)**

Special cases

Functions that can be expressed as special cases of the confluent hypergeometric function include:

- **Bateman's function**
- **Bessel functions and many related functions such as Airy functions, Kelvin functions, Hankel functions.**

  For example, the special case *b* = 2*a* the function reduces to a Bessel function:

$$_1F_1(a,2a,x) = e^{\frac{x}{2}}\,{}_0F_1(;a+\tfrac{1}{2};\tfrac{1}{16}x^2)$$

$$= e^{\frac{x}{2}}\left(\tfrac{1}{4}x\right)^{\frac{1}{2}-a}\Gamma\left(a+\tfrac{1}{2}\right)I_{a-\frac{1}{2}}\left(\tfrac{1}{2}x\right).$$

  This identity is sometimes also referred to as Kummer's second transformation. Similarly

$$U(a,2a,x) = \frac{e^{\frac{x}{2}}}{\sqrt{\pi}}x^{\frac{1}{2}-a}K_{a-\frac{1}{2}}\left(\frac{x}{2}\right),$$

  where *K* is related to Bessel polynomial for integer *a*.

- **The error function can be expressed as**

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2x}{\sqrt{\pi}}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2},-x^2\right).$$

- **Coulomb wave function**
- **Cunningham functions**

- **Elementary functions such as sin, cos, exp. For example,**
  $U(-n,1-n,x)=x^n$
- **Exponential integral and related functions such as the sine integral, logarithmic integral**
- **Hermite polynomials**
- **Incomplete gamma function**
- **Laguerre polynomials**
- **Parabolic cylinder function (or Weber function)**
- **Poisson–Charlier function**
- **Toronto functions**
- **Whittaker functions $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ are solutions of Whittaker's equation that can be expressed in terms of Kummer functions M and U by**

$$M_{\kappa,\mu}\left(z\right)=\exp\left(-z/2\right)z^{\mu+\frac{1}{2}}M\left(\mu-\kappa+\frac{1}{2},1+2\mu;z\right)$$
$$W_{\kappa,\mu}\left(z\right)=\exp\left(-z/2\right)z^{\mu+\frac{1}{2}}U\left(\mu-\kappa+\frac{1}{2},1+2\mu;z\right)$$

- **The general p-*th* raw moment (p not necessarily an integer) can be expressed as**

$$\mathrm{E}\left[\left|N\left(\mu,\sigma^2\right)\right|^p\right]=\left(2\sigma^2\right)^{\frac{p}{2}}\frac{\Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi}}\,{}_1F_1\left(-\frac{p}{2},\frac{1}{2},-\frac{\mu^2}{2\sigma^2}\right),$$
$$\mathrm{E}\left[N\left(\mu,\sigma^2\right)^p\right]=\left(-2\sigma^2\right)^{\frac{p}{2}}\cdot U\left(-\frac{p}{2},\frac{1}{2},-\frac{\mu^2}{2\sigma^2}\right)$$
(the function's second branch cut can be chosen by multiplying with $(-1)^p$).

Application to continued fractions

By applying a limiting argument to Gauss's continued fraction it can be shown that:

$$\frac{M(a+1,b+1,z)}{M(a,b,z)}=\cfrac{1}{1-\cfrac{\frac{b-a}{b(b+1)}z}{1+\cfrac{\frac{a+1}{(b+1)(b+2)}z}{1-\cfrac{\frac{b-a+1}{(b+2)(b+3)}z}{1+\cfrac{\frac{a+2}{(b+3)(b+4)}z}{1-\ddots}}}}}$$

and that this continued fraction converges uniformly to a meromorphic function of *z* in every bounded domain that does not include a pole.

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